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MECHANICS

AND

HYDROSTATICS

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MECHANICS

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MECHANICS

AN ELEMENTARY TEXT-BOOK

THEORETICAL AND PRACTICAL

BY

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PREFACE.

IT has now come to be generally recognized that the most satisfactory method of teaching the Natural Sciences is by experiments which can be performed by the learners themselves. In consequence many teachers have arranged for their pupils courses of practical instruction designed to illustrate the fundamental principles of the subject they teach. The portions of the following book designated EXPERIMENTS have for the most part been in use for some time as a Practical Course for Medical Students at the Cavendish Laboratory.

The rest of the book contains the explanation of the theory of those experiments, and an account of the deductions from them. This part has grown out of my lectures to the same class. It has been my object in the lectures to avoid elaborate apparatus and to make the whole as simple as possible. Most of the lecture experiments are performed with the apparatus which is afterwards used by the class, and whenever it can be done the theoretical

consequences are deduced from the results of these experiments.

In order to deal with classes of considerable size it is necessary to multiply the apparatus to a large extent. The students usually work in pairs and each pair has a separate table. On this table are placed all the apparatus for the experiments which are to be performed. Thus for a class of 20 there would be 10 tables and 10 specimens of each of the pieces of apparatus. With some of the more elaborate experiments this plan is not possible. For them the class is taken in groups of five or six, the demonstrator in charge performs the necessary operations and makes the observations, the class work out the results for themselves.

It is with the hope of extending some such system as this in Colleges and Schools that I have undertaken the publication of the present book and others of the Series. My own experience has shewn the advantages of such a plan, and I know that that experience is shared by other teachers. The practical work interests the student. The apparatus required is simple; much of it might be made with a little assistance by the pupils themselves. Any good-sized room will serve as the Laboratory. Gas should be laid on to each table, and there should be a convenient water supply accessible; no other special preparation is necessary.

The plan of the book will, I hope, be sufficiently clear; the subject-matter of the various Sections is indicated by the headings in Clarendon type; the Experiments to be performed by the pupils are shewn thus:

EXPERIMENT (1). *To explain the use of a Vernier and to determine the number of centimetres in half a yard.*

These are numbered consecutively. Occasionally an account of additional experiments, to be performed with the same apparatus, is added in small type. Besides this the small-type articles contain some numerical examples worked out, and, in many cases, a notice of the principal sources of error in the experiments, with indications of the method of making the necessary corrections. These latter portions may often with advantage be omitted on first reading. Articles or Chapters of a more advanced character, which may also at first be omitted, are marked with an asterisk.

I have found it convenient when arranging my own classes to begin with a few simple measurements of length, surface, volume and the like. These are given in Chapter I.

The two following chapters deal with Kinematics and treat the subject in the usual method.

When questions dealing with Momentum, Force, and Energy come to be considered two courses at least are open to the teacher. It is possible to make the whole subject purely deductive; we may start with some definitions and axioms—laws of motion, either as Newton gave them, or in some modern dress—and from these laws may deduce the behaviour of bodies under various circumstances.

Another and more instructive method, it seems to me, is to attempt to follow the track of the founders of Mechanics, to examine the circumstances of the motion of bodies in certain simple cases in the endeavour to discover the laws to which they are subject. This method has been followed in Chapters IV. and V. I have made

free use of a piece of apparatus—the ballistic balance—devised by Professor Hicks of Sheffield, and by its aid the student is led to realize the importance of momentum in dynamics and to study the transference of this quantity from one body to another. The rate at which momentum is transferred is then considered (Chapter V.) and a name—Force—is given to the rate of transference. It is shewn that in many cases the rate of change of momentum is constant; while others are referred to in which the rate of change of momentum depends only on the position of the body. Experiments are described to prove that in a given locality all bodies fall with the same uniform acceleration.

It is then shewn that with Atwood's machine, when the rider is on, the weights move with uniform acceleration; and hence the kinematical formulæ obtained earlier in the book relating to the motion of a particle moving with uniform acceleration are verified by experiment; the connexion between the mass moved, the acceleration and the weight of the rider is also investigated.

Some idea of the laws of motion in a simple case having been thus obtained from observation and experiment, Newton's Laws of Motion are enunciated in Chapter VI. and their consequences are deduced in the ordinary way. Some portions of the preceding chapters are of necessity repeated by this method of procedure, which may have other disadvantages as well. These I hope are counter-balanced by the gain resulting from a more intelligent appreciation of the subject on the part of the learner.

Mechanics is too often taught as a branch of pure

mathematics. If the student can be led up to see in its fundamental principles a development of the consequences of measurements he has made himself, his interest in his work is at once aroused, he is taught to think about the physical meaning of the various steps he takes and not merely to employ certain rules and formulæ in order to solve a problem.

Chapter VIII. deals with the third law of motion and the principle of energy; while in the succeeding chapters other problems are discussed.

The Second Part of the book deals with Statics. I believe it to be desirable that a student should commence the study of Mechanics with Kinematics and Kinetics, and have therefore arranged the book on this plan. At the same time it will, I hope, be found that the Statics is in great measure independent of the other part of the subject, though at the cost of some repetition. It will be possible therefore for a teacher to take it before the Kinematics.

In the Third Part some of the simple experimental laws of Hydrostatics are discussed and explained.

The book has grown considerably beyond the limits of my lectures, though it is by no means a complete treatise on Elementary Mechanics; still I hope it may prove useful as an introduction to the subject.

I have to thank many friends for help. Mr Wilberforce and Mr Fitzpatrick have assisted in arranging and devising many of the experiments. Mr Fitzpatrick has also read all the proofs. Dr Ward's suggestions in many parts of Chapters IV., V., VI. and VII. have been of the highest value. My pupil, Mr G. G. Schott of Trinity

College, collected for me many of the Examples, while Mr Green of Sidney College has most kindly worked through all the Examples and furnished me with the answers.

The illustrations have for the most part been drawn by Mr Hayles from the apparatus used in the class.

R. T. GLAZEBROOK.

CAVENDISH LABORATORY.
January, 1895.

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DYNAMICS

DYNAMICS.

CHAPTER I.

FUNDAMENTAL QUANTITIES. METHODS OF MEASUREMENT.

1. Mechanics and its signification. Mechanics is defined by Kirchhoff as the Science of Motion. Its object is to describe the kinds of motion which occur in Nature completely and in the simplest manner.

Motion is change of position; that which moves is known as "Matter."

For the complete apprehension of Mechanics the ideas of **Space, Time** and **Mass** are necessary and sufficient.

To these fundamental notions the idea of **Force** depending on the mutual action of bodies is subsidiary.

It will be our first step to consider in some detail the various quantities with which we have to deal and the methods we employ to measure them.

2. Units of Measurement. Any Quantity is of necessity measured in terms of a unit of its own kind; thus we measure the distance between two points in miles or feet, centimetres or inches; the area of a field in square yards or square metres, the mass of a lump of stone in tons or kilogrammes, the time between two events in hours or seconds. We shall thus have to consider, firstly, what are the units in terms of which the various quantities which occur in

Mechanics are to be measured, and, secondly, how we shall compare the quantities with these units.

Thus, for example, when it is stated that the distance between two fixed points is three feet, it is implied that a certain unit of length called a "foot" has been adopted and that three of these placed end to end exactly cover the distance.

3. Fundamental Quantities in Mechanics. The three fundamental quantities in Mechanics in terms of which other quantities which may occur can be measured are, **Length, Time** and **Mass.**

4. The Unit of Length. The unit of length generally used in England is the **Yard.**

Other measures of length, the inch, foot, fathom, mile, etc. are submultiples or multiples of the yard, and can be expressed in terms of it.

The **Yard** is defined by Act of Parliament[1] as follows: "The straight line or distance between the centres of the transverse lines on the two gold plugs in the bronze bar deposited in the office of the Exchequer shall be the genuine standard yard at 62° Fahrenheit, and if lost it shall be replaced by its copies."

In accordance with the Weights and Measures Act of 1878 the British Standards are now kept at the Standards' Office of the Board of Trade at Westminster. The copies referred to above are those preserved at the Royal Mint, the Royal Society, the Royal Observatory, and the Houses of Parliament.

Another unit of length which is authorized by Act of Parliament in England is the **Metre.**

This was defined by a law of the French Republic in 1795 to be the distance between the ends of a rod of platinum made by Borda, the temperature of the rod being that of melting ice. At this date the distance between the pole and the equator along a certain meridian arc of the Earth's surface

[1] 18 and 19 Vict. cap. 72, July 30, 1855.

had recently been measured by Delambre, and it was supposed that Borda's platinum rod represented one ten-millionth of this distance.

Further research has shewn that this is not exactly the case, and thus the metric standard of length is not the terrestrial globe but Borda's platinum rod.

The divisions of the metre are decimal. Thus

10 Decimetres = 1 Metre.
10 Centimetres = 1 Decimetre.
10 Millimetres = 1 Centimetre.

It is this fact and not the actual length which gives the metric system its value for scientific measurements. In such measurements the unit of length is now almost invariably the **Centimetre,** that is to say, it is one-hundredth part of the length of Borda's platinum rod when at the temperature of melting ice.

The relation between these two standards, the yard and the metre, has been the subject of very careful investigation. According to the most recent measurements it has been found that

$$1 \text{ metre} = 1{\cdot}09362 \text{ yards}$$
$$= 39{\cdot}37079 \text{ inches.}$$

Hence

$$1 \text{ inch} = 0{\cdot}0253995 \text{ metre}$$
$$= 2{\cdot}53995 \text{ centimetres.}$$

In this book we shall adopt the **Centimetre** as the unit of length.

5. Methods of measuring Lengths. The measurement of a length consists in the determination of the number of centimetres and fractions of a centimetre which are contained in it, and the method to be adopted in making the measurement will depend to some extent on the magnitude of the length; different methods would be required to measure a fraction of a centimetre or many kilometres. Some of the methods used in measuring small lengths will be given later.

6. Measurement of Area and of Volume. The units of area and of volume depend directly on that of length; they are respectively a square whose side is one centimetre, and a cube whose edge is one centimetre: in measuring an area we determine the number of square centimetres it contains, in measuring a volume we find the number of cubic centimetres in it. The volume of 1000 cubic centimetres is called a Litre, and is often employed as a unit of volume.

7. Experiments on Measurement of Length, Area and Volume.

EXPERIMENT 1. *To explain the use of a vernier and determine the number of centimetres in half a yard.*

You are given a rod half a yard long and a metre scale divided to centimetres. The scale has a vernier attached. This is another short loose scale which has ten divisions marked on it. On laying this along the metre scale it will be found (as in Fig. 1) that these ten divisions occupy the

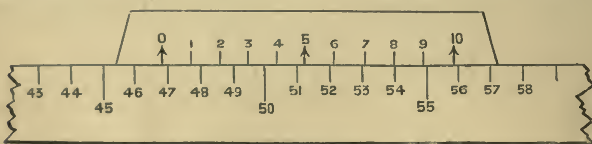


Fig. 1.

same length as nine divisions of the scale. Each division is therefore $\frac{9}{10}$ of a division of the scale, so that one division of the vernier is less than one of the scale by $\frac{1}{10}$ of a scale division, that is, in this case, by 1 millimetre. Place the rod so that one end coincides exactly with the end division of the scale, and place the vernier along the scale so that its end division, marked by an arrow, coincides with the other end of the rod. This division will probably not come opposite to a division of the scale. Suppose it falls between two, say 46 and 47. The rod is between 46 and 47 centimetres long. The vernier enables us to measure the exact length more

nearly; for on looking along the vernier it will be seen that its divisions and those of the scale get more and more nearly coincident until some division of the vernier, say the eighth, coincides almost exactly with one of the scale. Let us count back from this to the arrow-head or division 0 of the vernier, remembering that a vernier division is 1 mm. less than a scale division. The distance between 7 of the vernier and the corresponding scale division is 1 mm., between 6 and the scale division 2 mm., and so on, so that the distance between 0 and the scale division, which we have supposed to be 46, is 8 mm. The rod is therefore 46·8 cm. long. Had the coincidence of vernier and scale been at 5 or 6 the rod would have been 46·5 or 46·6 cm. We have merely to note the division of the vernier which coincides with a scale division and remember that in this case, when 10 vernier divisions coincide with 9 scale divisions, the divisions of the vernier enable us to read to tenths of the scale divisions. Other examples of the vernier should be studied, such as one in which twenty divisions correspond to nineteen of the scale, which therefore reads to twentieths of a scale division.

EXPERIMENT 2. *To find the circumference of a circular disc and so to verify the formula* Circumference $= 2\pi \times$ Radius *where π stands for $3\frac{1}{7}$ approximately.*

Measure the diameter of the disc by laying it on a finely divided scale, or better by the use of the calipers (Fig. 2)—a

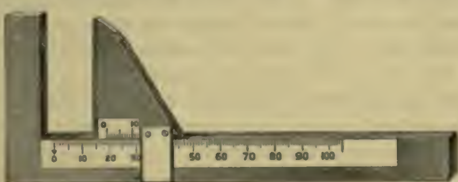


Fig. 2.

pair of calipers for the purpose can be constructed out of a draughtsman's T-square and set square. A scale is marked along the straight edge of the T-square and a vernier on the set square. Lay the T-square on the table and place the

disc so as to touch both the straight edge and the cross piece of the square. Place the set square against the straight edge and slide it along until it touches the disc. Since now both the cross piece of the T-square and one edge of the set square are at right angles to the straight edge of the former the distance between these two as measured along the scale and vernier will give the diameter of the disc. To find the circumference; make a mark with a pencil, or otherwise, on the edge of the disc, and place this in contact with the zero of a finely divided scale, then roll the disc along the scale, taking care that it does not slide, until the mark again comes in contact with a division of the scale. Note this division. The distance between this division and the zero of the scale is equal to the circumference of the disc. Repeat the observations to secure accuracy. It will be found that the ratio of the circumference to the radius is approximately equal to $2 \times \frac{22}{7}$. This ratio is usually denoted by 2π. Thus $\pi = \frac{22}{7}$ approximately.

EXPERIMENT 3. *To find the area of a circle and to verify the formula* Area $= \pi r^2$, *where* r *denotes the radius of the circle.*

You are given a sheet of paper divided by two series of parallel lines at right angles to each other into a number of small squares. The distance between any two consecutive lines is $\frac{1}{10}$ inch, so that each square has an area of ·01 sq. inch. Draw on this a circle of some 3 or 4 inches diameter and measure the diameter. The circle will enclose a large number of complete squares. Count these up and reckon their area. The circumference will also intersect a number of squares. Estimate for such intersected squares the total area which lies within the circle. Thus the area of the circle can be found approximately and the formula verified. This method can be applied to any other figure.

EXPERIMENT 4. *To find the volume of a sphere and to verify the formula* Volume $= \frac{4}{3}\pi r^3$, *where* r *is the radius of the sphere.*

Measure the diameter of the sphere by the calipers and hence find its radius. Place the sphere in a test tube or small beaker, Fig. 3, which has a mark made on its outside by means of a file or by gumming on a piece of paper. Fill a burette with water up to a known volume, and let the water run from the burette into the beaker until the latter is filled up to the mark, and note at what level the water in the burette now stands. Find hence the volume of water which has been placed in the beaker. Remove the sphere and the water from the beaker, and by again letting water run in from the burette find the volume of the beaker up to the mark. The difference between these two volumes is clearly the volume of the sphere, and the formula can be verified. The volume of any other solid which sinks in water can be found in the same way. Care must be taken to remove all air bubbles from the solid.

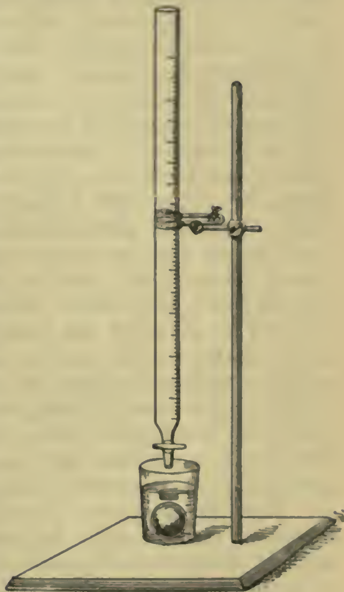


Fig. 3.

EXPERIMENT 5. *To find the thickness of a glass cover-slip by the screw.*

The instrument, Fig. 4, consists of a platform with three feet, whose extremities form an equilateral triangle. Through the centre of the platform passes a fourth foot, which can be raised or lowered by means of a screw.

The pitch of the screw used is 20 threads to the inch, so that if the platform in which the screw works be held firm and the screw turned once round, its end advances or recedes $\frac{1}{20}$ of

an inch. A disc is fixed on to the screw-head and its edge divided into 100 parts, and a vertical scale divided into $\frac{1}{20}$ inch is attached to the platform. If the disc be turned so as to bring the edge of this scale from any one division to the next, the end of the screw moves one-hundredth of one-twentieth of an inch, or $\frac{1}{2000}$ inch; hence, by noting the number of whole turns and parts of a turn made by the disc, we can measure the distance moved over by the screw-point. The whole number of turns are given by the readings of the vertical scale, for the disc moves over one division for each turn. Place the instrument on a flat sheet of glass, and turn the disc until the screw-point is in contact with the glass, read the screw-head; place five or six cover-slips one on the top of the other on the glass and raise the screw until they will just pass underneath it. Read the position of the disc when the point of the screw is just in contact with the top cover-slip, having noted the number of whole turns made by it. From this whole number of turns and the two readings of the disc calculate the total thickness of all the cover-slips, and then, by dividing this by the number of slips, the average thickness of each one.

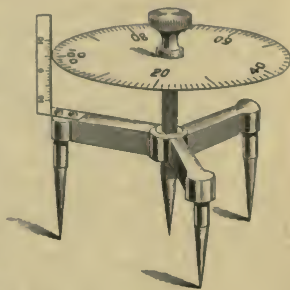


Fig. 4.

EXPERIMENT 6. *To use the screw gauge to measure the diameter of a wire.*

The Screw Gauge is shewn in Fig. 5. It consists of a metal arm ABC; through one end of this passes a steel plug D with a planed face, and through the other a screw EF. The pitch of the screw is half a millimetre, and the end E is planed so as to be parallel to the

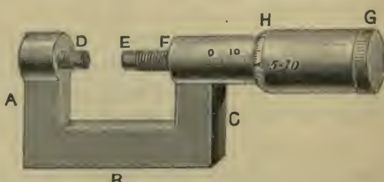


Fig. 5.

face D and perpendicular to the axis of the screw. The screw can be turned by means of the milled head G until its end E comes in contact with D; each complete turn of the screw separates the planed faces by half a millimetre. A scale of half-millimetres is engraved on the frame of the instrument parallel to the axis of the screw and the milled head G carries a cap H with a bevelled edge. The circumference of this edge is divided into a scale of fifty parts, and when the end of the screw is in contact with D the zero of this scale and the zero of the scale on the frame should coincide. On making a complete turn of the screw the cap is moved back half a millimetre, and the zero mark on the bevelled edge is brought opposite the first division of the linear scale. Thus the divisions of this scale which are exposed register the number of half-millimetres between D and E. Since the bevelled edge is divided into fifty parts a rotation through a single part corresponds to a separation of the plane ends by $\frac{1}{50}$ of $\frac{1}{2}$ of a millimetre or by $\frac{1}{100}$ of a millimetre. Thus, if a division (say 24) of the bevelled edge coincides with the linear scale, the distance between the plane faces is a whole number of half-millimetres, which is given by the number of divisions of the linear scale exposed, together with $\frac{24}{100}$ of a millimetre. Thus, to measure the distance between the plane ends, read the number of half-millimetres exposed on the linear scale and add to this the number of hundredths of a millimetre given by the reading of the scale on the bevelled edge.

When using the instrument to measure the diameter of a wire first test the zero reading; then hold the wire between D and E and turn the screw-head G until the wire is gently clipped between the two plane faces. In this case the distance between these faces is the diameter of the wire.

8. Other Instruments for measuring lengths.

(a) Scales and Compasses.

A pair of compasses and a finely divided scale are often the most convenient apparatus for measuring lengths. The compasses are adjusted until the distance between their points is exactly the length to be measured; they are then applied to the scale and the length is read off. Instead of an ordinary scale a diagonal scale may be used. This is shewn in Fig. 6. There are eleven equidistant parallel lines running the whole length of the scale dividing it into ten spaces. The scale is

divided into inches by lines running across it at right angles to this series of parallel lines. These are numbered **0, 1, 2, 3**, starting from *D* as zero. The first inch, *AB* and *CD*, along each of the top and bottom lines

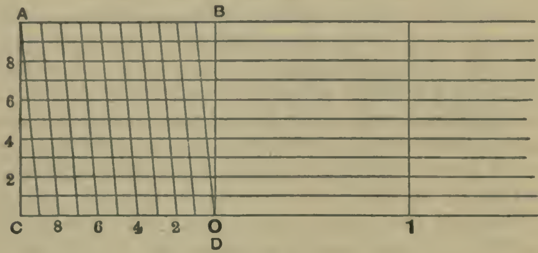


Fig. 6.

is divided into tenths. The alternate divisions, starting from *D* to the left, are numbered 2, 4, 6 etc., the alternate vertical divisions starting from *C* upwards are also numbered. Lines are drawn obliquely across the first rectangular division *ABDC* of the scale as shewn in the figure; thus *A* is joined to division 9 of *CD*, division 9 of *BA* to division 8 of *CD* and so on; these oblique lines enable us to measure to one-tenth of the small divisions of the scale. For consider the distance between any vertical line, say that through the **1** inch division, and an oblique line such as that through the small division 4. When measured along the lowest horizontal line this distance is 1·4 inches, when measured along the top line it is 1·5 inches. Thus on passing from the bottom to the top of the scale it increases by ·1 inch, but it increases by an equal amount for each vertical space passed over, and here are ten of these spaces, hence the increase for each vertical space is ·01 inch. Thus the distance along the fifth line from the bottom and between the vertical line through the 1 inch mark and the oblique line through the ·4 inch mark is 1 inch + 4 tenths + 5 hundredths or 1·45 inches.

Thus to measure on the scale a distance with the compasses place the right leg on one of the vertical divisions at the point where it crosses the bottom horizontal line, say at division **2** inch. Let the left leg of the compass fall between the points in which the fourth and fifth oblique lines cut the bottom horizontal line. Then the distance is between 2·4 and 2·5 inches. Slide the compasses upwards, keeping the right-hand leg in the vertical division through **2**, and the line joining the two legs parallel to the horizontal lines on the scale, until the left-hand leg falls on a horizontal line as close as possible to the point in which it is intersected by one of the oblique lines; let this occur on the fifth horizontal line; the distance between the legs is greater than 2·4 inches by 5 hundredths of an inch; thus it is 2·45 inches.

(*b*) Caliper-Compasses are made in special forms for measuring the dimensions of curved bodies. Thus, Fig. 7 shews a pair of Calipers of simpler construction than the slide calipers described in EXPERIMENT 2.

The *outside calipers AB* can be set so that the points A, B just come in contact with two points on the outside of a cylindrical or convex surface, the distance between which is required, while by means of the *inside calipers CD* the distance between points on the inside of a cavity within which the instrument can be introduced can be measured; in either case the distance is found by adjusting the calipers and then laying off the length between the points on a scale.

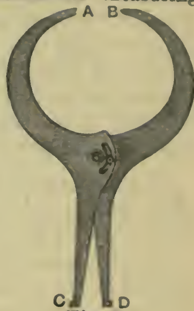


Fig. 7.

(*c*) *The Beam Compass.* This instrument is shewn in Figure 8. A sliding piece C, fitted with a vernier and a clamping screw, is attached

Fig. 8.

to a long straight scale AB. A point D is attached to one end A of the scale and the sliding piece C carries a similar point E.

The instrument is adjusted so that when the points D and E are in contact the vernier is at zero on the scale; the reading then of the scale and vernier in any position gives the distance between the points D and E. The instrument is set so that D and E coincide respectively with the two points the distance between which is required, and this distance can then be read off directly[1].

9. Time. The next fundamental Physical Quantity which we have to consider is **Time**. "The idea of Time," says Maxwell, "in its most primitive form is probably the recognition of an order of sequence in our states of consciousness." We can

[1] For further particulars as to the method of using such measuring instruments, see Glazebrook and Shaw, *Practical Physics*, Chapter IV.

associate certain sense expressions in a group and separate them off from other groups which we perceive *simultaneously*. Thus we gain the idea of space, but we have also the power of perceiving things *in succession*, we recognize a group of sense impressions as like another distinct group, the impress of which is stored in our memory; we perceive events which follow each other as well as others which have a simultaneous existence.

Our measure of time is derived from the apparent motion of the stars; this apparent motion is a consequence of the motion of the earth round its axis, and we feel that this motion, in the interval from noon to noon, marks off series of like sequences of events; we recognize that the time occupied in one such complete rotation is approximately constant. Owing to the motion of the earth round the sun the interval between two successive passages of the sun across the meridian of any place differs slightly from day to day. The average of such intervals during the year is the mean solar day.

A mean solar day contains 86400 seconds and the fundamental unit of time is the **Mean Solar Second.**

10. Mass. Our third fundamental quantity is called Mass.

If we consider the bodies with which we have to deal as composed of Matter, then any body will consist of a definite quantity of matter. This quantity is usually called its **Mass.**

We shall find however in the sequel that we can give a definite meaning to the term Mass as used in Mechanics without attempting to define the term Matter. We have means for comparing with great accuracy the masses of different bodies; we can therefore measure the Mass of any body in terms of some standard Mass. For the present then we look upon **Mass** as a property of bodies which we recognize by experiment and which we can only define when we have considered those experiments.

We do not know what matter is, it may be that it has no phenomenal existence apart from our conception of it, but it is beyond our province to discuss this here. If we assume that there is a substratum of something we call matter in a body, then the quantity of that matter is measured by the mass of the body, and the masses of bodies can be compared in an exact manner.

There is one case in which there is no difficulty in comparing the quantity of matter in two bodies. For consider two cubes of some homogeneous[1] substance such as platinum, each one centimetre in edge, they are alike in all respects; if they are composed of matter the quantities they contain are obviously equal. The two cubes combined will have double the volume of either singly; there is double the quantity of matter in the two that there was in either. *The quantities of matter in two portions of the same homogeneous substance are proportional to the volumes of the two.* We cannot apply this argument to portions of different substances; equal volumes of iron and lead we shall see have different masses, they are said to contain different "quantities of matter"; it is only when we have considered the laws of motion that we can state exactly what is meant when we assert that the "quantities of matter" in two given bodies are equal, and how it is possible to compare the "quantity of matter" in a lump of iron with that in a heap of feathers.

11. Measurement of Mass. Masses are measured in terms of a unit of mass.

DEFINITIONS. *The mass of a certain lump of Platinum marked PS* 1844 1 *lb., deposited in the Standards department of the Board of Trade at Westminster, is the English Unit of Mass and is called the* Pound Avoirdupois.

The mass of a certain lump of platinum made by Borda in 1795 *and kept at Paris is the Unit of Mass on the metrical system and is called the* Kilogramme.

It was intended that Borda's kilogramme should be equal to the mass of 1000 cubic centimetres of distilled water at 4° C.[2] The exact determination of the mass of such a volume of water is difficult and its value probably differs slightly from that of Borda's platinum mass. Hence the

[1] A *homogeneous* substance is one which has identical properties at all points. Water or any other liquid, glass, brass, iron are examples of homogeneous substances. Substances, such as a piece of conglomerate rock, which have different properties at different points, are called *heterogeneous.*

[2] The mass of water which can be contained in 1000 c.c. is greater at this temperature than at any other, hence this temperature was chosen as the standard. See Glazebrook, *Cambridge Natural Science Manuals, Heat*, p. 88.

metrical standard of mass is really the mass of a lump of platinum and not, as was intended, the mass of a definite number of units of volume of water. The volume at 4° C. of a kilogramme of water is called a Litre and is equal to 999·97 cubic centimetres.

Still the statement that the mass of 1 cubic decimetre (1000 c.c.) of distilled water is 1 kilogramme is sufficiently nearly true for most purposes, and enables us to introduce great simplification into many numerical calculations.

The unit of mass which is now usually adopted for scientific purposes is the **Gramme**. **One Gramme** *contains one-thousandth part of the mass of Borda's kilogramme Standard.*

Since a kilogramme (1000 grammes) is very approximately the mass of 1000 c.c. of distilled water, a gramme is very approximately the mass of 1 c.c. of distilled water at 4° C.

The divisions of the gramme are decimal:

10 Decigrammes = 1 Gramme,
10 Centigrammes = 1 Decigramme,
10 Milligrammes = 1 Centigramme.

Careful experiment has shewn that the kilogramme contains 2·2046 pounds.

12. Density. For a given substance, the mass of a body depends on its volume, while, for bodies of given volume, the mass depends on the substance of which the bodies consist and on its physical state. A large lump of iron has a greater mass than a small lump of iron, but a small lump of iron may be of greater mass than a large lump of cork. It is useful to have some term to denote the mass of a definite volume of any body, say 1 cubic centimetre.

DEFINITION OF DENSITY. *The* **Density** *of any homogeneous substance is the mass of unit volume of that substance.*

It follows from this definition that to determine the density of a body we must find the number of units of mass in the unit of volume, we require therefore to know the unit of mass and the unit of volume, if these be the gramme and the cubic centimetre respectively, we may say that the density is so

many grammes per cubic centimetre. Thus in these units the density of water is 1 gramme per c.c., that of iron 7·76 grammes per c.c. In any other units the numerical measures of the densities of these substances would differ from the above. Thus a cubic foot of water contains 998·8 oz. or 62·321 lb.; hence the density of water is 998·8 oz. per cubic foot or 62·321 lb. per cubic foot; iron is 7·76 times as dense as water, hence its density is 7·76 × 62·321 lb. per cubic foot.

From the above definition of density we can find a relation between the mass, the volume and the density of a body.

PROPOSITION 1. *To shew that if the mass of a homogeneous body be* M *grammes, its density* ρ *grammes per cubic centimetre and its volume* V *cubic centimetres, then* $M = V\rho$.

For by the definition,

$$\begin{aligned} \text{the mass of 1 c.c.} &= \rho \text{ grammes,} \\ \text{therefore the mass of 2 c.c.} &= 2\rho \text{ grammes,} \\ \text{and the mass of 3 c.c.} &= 3\rho \text{ grammes,} \\ \text{hence the mass of } V \text{ c.c.} &= V\rho \text{ grammes.} \end{aligned}$$

Therefore $$M = V\rho.$$

We may write this as

$$\rho = \frac{M}{V},$$

and thus we have the result that the density of a homogeneous substance is the ratio of its mass to its volume.

A result similar to the above holds for any other consistent system of units.

Various methods of determining by experiment the density of a body will be given later[1].

13. The Comparison of Masses. A balance is the instrument usually employed in the comparison of masses. The theory of the balance is discussed in the Statics, and the method of measuring mass is there considered.

[1] See Hydrostatics.

14. The C.G.S. system of measurement. We shall find that the other physical quantities with which we have to deal in Mechanics can be expressed in terms of the units of length, time and mass or of some of these units. When we take the Centimetre, the Gramme and the Second as fundamental units we are said to employ the C.G.S. system. This is now generally used for scientific purposes; when a quantity has been measured in terms of these fundamental units, it is said to have been determined in absolute measure.

If we know the relation between the C.G.S. system and some other system of units it is easy to change from one to the other in our calculations. Thus if we wish to change to the Foot-Pound-Second System we have approximately

$$1 \text{ cm.} = \cdot 03281 \text{ feet.}$$
$$1 \text{ gramme} = \cdot 002205 \text{ pound.}$$

Examples. (1). *Find the number of cubic feet in* 1000 *cubic centimetres.*

$$1000 \text{ c.cm.} = 1000 \times (\cdot 03281)^3 \text{ c. feet}$$
$$= \cdot 03532 \text{ c. feet.}$$

(2). *The density of a piece of glass is* 2·5 *grammes per c.cm., find it in lb. per c. foot.*

We have

$$1 \text{ c.cm.} = (\cdot 03281)^3 \text{ c. feet}$$
$$= \cdot 00003532 \text{ c. feet}$$
$$2\cdot 5 \text{ grammes} = 2\cdot 5 \times \cdot 002205 \text{ lb.}$$

Hence a volume of ·00003532 c. feet contains 2·5 × ·002205 lb.

Thus density required

$$= \frac{4\cdot 5 \times \cdot 002205}{\cdot 00003532} \text{ lb. per c. foot.}$$

And this reduces to 2·5 × 62·43 or 156·08 lb. per c. foot.

15. Terms used in Mechanics. Mechanics is the Science of Motion. In studying motion we shall generally require to know both the **Displacement** or change in position of the body, and also the time during which that displacement has occurred.

This branch of the subject is called **Kinematics.** It may be described as the Geometry of Motion; Geometry deals with Space only, Kinematics has for its subject Space and Time.

When we come to consider the mutual relations between moving bodies the science of motion is called **Kinetics**; while in **Dynamics** we pay special attention to the connexion between Force and Motion. In **Statics** we consider the conditions which must exist among a set of Forces impressed on a body which remains at rest.

Statics and Dynamics are usually applied to the Mechanics of Solid bodies. The Sciences which deal with the equilibrium and motion of Fluid bodies are respectively **Hydrostatics** and **Hydrodynamics.**

We are concerned in nature with Material Bodies. A **Body** is a portion of "matter" bounded in every direction. We shall consider a Body as composed of a number of material Particles.

A Material Particle is a portion of "matter" so small that for the purposes of our investigations the distances between its different parts may be neglected.

In Dynamics we deal first with the motion of one or more isolated particles, or of a body which we can treat as a particle; we can afterwards proceed to consider the motion of a body of finite size.

We must remember that it will depend on circumstances whether we can treat a body as a particle or not. Thus, apart from a small effect due to the resistance of the air, the shape of a falling stone does not affect the rate at which it falls to the earth; we may solve a problem relating to a falling stone correctly on the supposition that the whole of the stone is concentrated into one point and that the stone behaves as a particle; the same would be true of a cricket ball so long as it is in the air, but the motion of the cricket ball, on striking the ground or being hit by the player, depends on its shape and on the amount of spin given to it by the bowler; these must be considered before we can state how it will move immediately after it is struck, in solving this part of the problem we cannot treat the ball as merely a particle. The words "for the purposes of our investigations" in the above definition are of importance; in the present book when considering Dynamics we deal with the motion of particles.

EXAMPLES.

MENSURATION.

[Take the value of π as 22/7.]

1. Reduce to centimetres (1) 1 ft. 2 in., (2) 2 yds., (3) 5 ft., (4) 1 furlong.

2. Reduce to kilometres (1) 1 mile, (2) 4000 miles, (3) 600 yds., (4) 1 metre, (5) 25 millimetres.

3. Find in centimetres the circumference of circles whose radii are (1) 1 ft., (2) 10 yds., (3) 4000 miles, (4) 750 metres.

4. Find in square centimetres the areas of the circles whose radii are given in Question 3.

5. A circle of radius 5 inches is cut out from a circular disc of radius 9 inches; find the area of the remainder.

6. The circumference of a circle is 1 mile; find its area.

7. The area of a circle is equal to that of a rectangle whose sides are 44 and 126 feet; find its radius.

8. A circle and a square have the same perimeter; determine which has the greater area.

9. An equilateral triangle is described on one side of a square of which the side is 10 feet; find the area of the figure thus formed.

10. A circle of 20 centimetres radius is divided into three parts of equal area by two concentric circles; find the radii of the circles.

11. A circle is 20 centimetres in radius; find the area of a square which can be inscribed in it.

12. A sphere when placed in a beaker as in Exp. 4 displaces 38·786 cubic centimetres of water; find its radius and its surface.

13. Ten cover-slips are placed under the spherometer as described in Experiment 5, the pitch of the screw being ½ a millimetre, and the point is raised 8·56 turns; find the average thickness of a cover-slip.

14. The density of copper is 8·95 grammes per c. cm. The diameter of a piece of copper wire is 1·25 mm. and its length 1025 cm.; find its mass.

15. Find the density of a cylinder 1 foot in height and 6 inches in radius whose mass is 60 lbs.

16. Find the density of a sphere 10 cm. in radius and 5 kilogrammes in mass.

17. Determine the density of the cylinder described in Question 15 in grammes per c. cm.

18. Find the density of a pyramid on a triangular base each side of which is 10 cm. and which has an altitude of 30 cm., the mass of the pyramid being 8 kilogrammes.

19. The density of mercury is 13·59 grammes per c. cm.; find it in grains per cubic inch.

20. Compare the densities of a sphere 5 cm. in radius, 5 kilos. in mass, and a cylinder 1 foot in height, 6 inches in radius and 60 lb. in mass.

CHAPTER II.

KINEMATICS. VELOCITY.

16. Motion. A Body is said to move when it is in different positions at different times. Thus, in order to determine the motion of a body, we have to determine its position at different times and investigate whether the position changes or not. We may notice first that the motion with which alone we can deal is relative motion.

Two passengers seated in a railway carriage are at rest relatively to each other and to the carriage; they are however in motion relatively to objects by the side of the line along which the train is moving. The planets are in motion relatively to the sun; the whole solar system, sun, planets and satellites, is in motion relatively to the stars. In all problems of motion we must have some point which so far as that motion is concerned we treat as fixed and from which we regard the motion. We may investigate the motion of a cricket ball thrown upwards from the earth, but in the investigation we should usually suppose the point from which the ball is thrown to be at rest; as a fact of course this is not true, that point and the ball with it partake of the motion of the earth round its centre, of the motion of the earth's centre round the sun, and so on, but for our purposes this is immaterial.

DEFINITION. **Motion** *is change of position.*

17. Measurement of Position of a Particle. The position of one particle relatively to another is determined

if we know the *length* and the *direction* of the line joining the two. We may say then that one particle, A Fig. 9, is in motion relatively to a second particle B whenever the length or direction of the line AB joining the two varies.

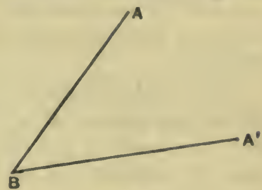


Fig. 9.

When a particle is moved from one position A to another position A' it is said to be displaced. The Displacement of the particle is measured by the length and direction of the line AA'.

We are however in general concerned with the rapidity with which the change in position occurs; moreover the motion may be *Uniform* or *Variable.*

DEFINITION. *The Motion of a particle is* Uniform *if the particle passes over equal spaces in equal times.*

The motion of a particle is Variable *if the particle passes over unequal spaces in equal times.*

Since an interval of time is measured by the angle which the earth turns through about its axis in that interval, equal times are those in which the earth turns through equal angles. If then in a series of intervals in which the earth turns through equal angles a moving particle passes over equal distances the motion of the particle is uniform; if on the other hand the distances traversed by the particle are unequal, while the angles turned through by the earth are equal, then the motion is variable.

18. Rate of Change of a Quantity. The phrase, Rate of, is one which will often occur and which it is desirable to consider. By *rate of change* is meant generally the change in a quantity which takes place during some given interval of time adopted for convenience as the unit of time—or more exactly the ratio which that change bears to the interval of time during which it has occurred.

Thus the statement that the Rate of Interest is 3 per cent. per annum means that in *one year* a sum of £100 increases by £3. If this rate continues uniform, then in 10 years the increase on £100 will be 3×10 or £30; the total increase is obtained by multiplying the rate of interest by the time during

which interest has accrued. Again, if we know that in 5 years £100 has increased by £15 and that the rate of interest has been uniform, we infer that that rate is 15/5 or 3 per cent. per year.

We might speak in the same way of the rate of growth of the population of a town, meaning the increase in population per week or per day as the case may be, or of the daily rate of progress of a building, meaning the amount built in a day.

Thus we see (1) Any quantity varies uniformly when it increases—or decreases—by equal amounts in equal times, and (2) the rate of change of a quantity, which varies uniformly, is the ratio of the change in that quantity to the interval of time during which it has occurred; it is measured by the change which takes place in the unit of time.

19. Average Rate of Change. But Quantities do not always change *uniformly*. The amount of interest obtainable for a given sum may vary from day to day. The daily rate of growth of the population of a town will not be the same throughout the year; more children are born on some days than on others. In such cases we are often concerned with the **Average**[1] **Rate of Change.** This average rate of change is found by calculating the actual change during any time and dividing that by the time.

Thus the statement that during the year the average daily rate of increase in the population of London was 340, does not mean that 340 children were born on each day, but that during the year 340×365 or 124100 were born, so that the total increase during the year is the same as though 340 were born on each day. Or again, consider the case of a railway train which performs a journey of 42 miles in an hour. We should say that its average rate of motion is 42 miles an hour, but this does not imply that it is moving uniformly at the rate which would enable it to traverse the distance in this time; at times it moves more quickly, at the stations it is at rest for some few minutes, and on starting again its rate of motion is less than 42 miles per hour. We must distinguish

[1] The word average as used here has reference to time.

between its *average* rate of motion and its rate of motion *at* any instant of the hour.

20. Graphical Representation of Rate of change. Suppose now we represent on a diagram the two cases of uniform and variable motion thus. Draw a horizontal line, say 30 cm. long, to represent an hour; divide it into half centimetres so that each 5 mm. represents 1 minute, and at the end of each division erect a vertical line to represent the distance traversed up to the end of that minute. Then, in the case of uniform motion, if we represent 1 mile by 1 cm., since 42 miles per hour is the same as 42/60 or ·7 miles per 1 minute, the first vertical line will be ·7 cm. or 7 mm. long, the second will be twice this, the third three times, and so on; the vertical lines will increase uniformly in height, each will be 7 mm. higher than the preceding; a line joining these ends will be straight and will be uniformly inclined to the horizon. The figure obtained will be similar to that shewn in Fig. 10.

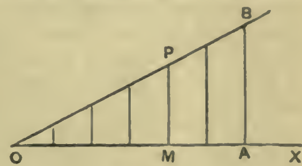


Fig. 10.

21. Variable Rate of Change.

Consider now the case in which the rate of motion is not uniform; let us suppose for the present that we may treat it as uniform during each minute, but that it varies from one minute to the next; the increments in the lengths of the various vertical lines will be different. The train starts slowly, the first line will therefore be less than 7 mm. long, the second less than 14 mm. After a time however the train must exceed its average rate of motion, each successive increment will be greater than 7 mm. until another station is approached, when they will again decrease; during the three or four minutes for which the train stops the corresponding vertical lines will all be of the same length, the

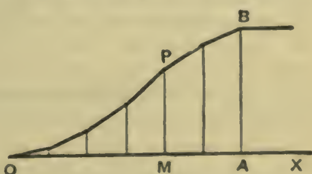


Fig. 11.

line joining their ends will be horizontal. The diagram obtained will resemble Fig. 11. The rate of change of the train's position during different minutes is variable; unequal distances are traversed in equal times; the rate of change however *for each minute* is obtained by finding the alteration during that minute; moreover, if we multiply the change so obtained by 60, the number of minutes in the hour, we can get the rate of motion in miles per hour.

Now the figure just obtained does not represent accurately the motion of the train, for it does not move uniformly for 1 minute, then change its rate of motion and so on; we should get a nearer representation to the truth if we divided each 5 mm. into 60 parts, each representing a second, and supposed the rate of motion to be uniform for each second but to change at the end of every second; this would not be exact, but by proceeding thus and dividing each second into a very large number of very small fractions and supposing the rate of motion uniform during each fraction, but variable from fraction to fraction, we may get as close a representation to the truth as we please. In this manner we come to see that *the Rate of Motion at any moment is found by dividing the distance traversed during an indefinitely short interval of time by the number of seconds in that interval.*

If we divide each minute as shewn in Figs. 10 or 11 into a very large number of parts we shall obtain, instead of the series of straight lines, such as those shewn in Fig. 11, a very much larger number of such straight lines. Each of these will be very short, and it will be impossible to distinguish the many-sided polygon thus formed and a figure bounded by a regular smooth curve as in Fig. 12. The one however represents the series of discontinuous changes at very brief intervals, the other the regular continuous change in the rate of motion which actually occurs with a train. We may compare the two processes with those of mounting a hill (a) by a series of stairs

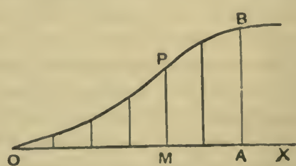


Fig. 12.

or steps, (*b*) by a gradual slope. If the number of steps in the stair be sufficiently large and the size of each sufficiently small the two processes are indistinguishable, the steps merge into the continuous slope. We arrive then at the definition of the term Rate of Change applicable to all cases.

DEFINITION OF RATE OF CHANGE. *The* Rate of Change *of any quantity is the ratio of the change in that quantity to the interval of time during which it has occurred when that interval is sufficiently small*[1].

The rate of change therefore is found by dividing the change which has taken place during an interval of time by that interval when the interval is made sufficiently small.

Let us now suppose that at a given moment the rate of change ceases to vary and that the quantity concerned continues to change at the same rate as it did at that moment. From this time on the rate of change is uniform, and is equal therefore to the ratio of the change in *any* interval—not necessarily a very small interval—to the interval during which that change has taken place. If we take the interval to be one second—the unit of time—we may in this case say that the rate of change is the change which has taken place in one second; or the change per second.

We may thus sum up with the statement that the *Rate of change of any quantity, when uniform, is the change in that quantity per second, and, when variable, is the ratio of the change which would take place in that quantity to the interval of time during which that change has occurred when that interval is made very small.* In measuring the rate of change of any quantity we adopt the process indicated in the definition above and divide the change occurring in any interval by the number of seconds in the interval when this number is sufficiently small.

22. Velocity and its measurement. We proceed now to apply the idea of rate of change to various dynamical quantities.

[1] This definition it will be seen is the same as the statement on page 21, except that the last clause is additional. This clause is not necessary if the rate of change be uniform.

DEFINITION. **Velocity** *is Rate of Change of Position.*

We have already defined Motion as change of position, we may therefore state that Velocity is rate of motion.

Velocity may be either *Uniform* or *Variable.* **Uniform Velocity** is measured by the change in position which occurs per second. **Variable Velocity** is measured by the ratio of the change in position in a given interval to the number of seconds in that interval when that number is sufficiently small, that is by the change which would occur per second if during the second the velocity remained uniform.

A particle moving with *uniform* velocity describes equal spaces in equal times. A particle moving with *variable* velocity describes unequal spaces in equal times.

To measure velocity we need to know the change in position, or displacement, per second; this change is determined if we know (1) the distance the particle moves through, (2) the direction of motion. The word **Speed** is employed to denote the distance traversed per second without reference to the direction.

DEFINITION. *The* **Speed** *of a particle is the rate at which it describes its path.*

A particle moves with uniform **Speed** if it travels over equal distances in equal intervals of time; if the **Velocity** be uniform, not merely will the distances be equal but their directions will be the same.

In strictness therefore the velocity of a particle can be uniform only when the particle is moving in a straight line and passes over equal distances in equal times. The term uniform velocity is however often applied where uniform speed would be more accurate; thus the hand of a clock is said to move with uniform velocity, but since the direction of motion changes the velocity is not strictly uniform though the speed is.

23. Motion of a particle with uniform speed.

PROPOSITION 2. *To find the distance—* s *cm.—traversed in* t *seconds by a particle moving with a uniform speed of* v *cm. per second.*

Since the speed is uniform, the particle passes over v cm. in each second.

Hence distance traversed in 1 second $= v$ cm.,
distance traversed in 2 seconds $= 2v$ cm.,
distance traversed in 3 seconds $= 3v$ cm.,
....
distance traversed in t seconds $= tv$ cm.

Therefore $$s = vt.$$

Hence also $$v = \frac{s}{t}.$$

This last result gives a formal proof of the statement that speed when uniform is measured *either* by the distance traversed per second *or* by dividing the distance traversed in any interval of time by that interval.

24. Average speed. The Average speed of a particle moving over a given distance in a given time may be defined as the speed with which a second particle, moving uniformly, would describe the given distance in the given time.

It is found therefore by dividing the distance traversed by the number of units of time taken to traverse it.

Thus, consider two trains, one of which is moving uniformly at the rate of 50 miles an hour, while the second starts from one station and arrives, after an interval of an hour, at a second 50 miles distant. The *average* speed of the second train is *the* speed of the first train.

25. Variable speed. The actual speed of the second train at each moment is not 50 miles an hour. To find its value we should require to determine the distance traversed by the train in some very short interval, and divide that distance by the interval; probably in the case of a train the speed would not vary much during a single second, and if we could measure accurately the distance traversed in one second we should determine the speed during that second.

26. Units of speed. Speed is measured by the number of units of length traversed per unit of time.

The numerical measure therefore of the speed or of the velocity of a particle will depend on the unit of length

and on the unit of time. The same speed may be expressed by very different numbers.

Thus, since a mile contains 5280 feet, a speed of 60 miles per hour is the same as one of 60×5280 feet per hour; and since an hour contains 3600 seconds this velocity is the same as one of $60 \times 5280/3600$, or 88, feet per second. In order then to specify a speed or a velocity we must state the unit of length and the unit of time.

A velocity of 88 means one which is 88 times as great as the unit of velocity, and conveys no definite information unless we state clearly what that unit is. It may be a velocity of 88 feet per second, or 88 miles per hour, or anything else.

DEFINITION. *A particle has* **Unit Velocity** *when it traverses unit distance in unit time.*

In the C.G.S. system, a particle has unit velocity when it traverses 1 centimetre per second.

In stating, then, that the velocity of a particle is v, it is necessary to specify the unit distance and the unit time, and to write, if the C.G.S. system be used, a velocity of v cm. per sec.

Examples. **(1).** *Which train has the greater speed, one moving at the rate of* 60 *miles an hour or one which travels* 100 *yards in three seconds?*

The first train in 60×60 seconds moves over

$$60 \times 1760 \text{ yards};$$

$$\therefore \text{ in 1 second it moves } \frac{1760 \times 60}{60 \times 60} \text{ yards};$$

$$\therefore \text{ the speed is } 29\tfrac{1}{3} \text{ yards per second.}$$

The second train in 1 second moves $\dfrac{100}{3}$ yards;

$$\therefore \text{ the speed is } 33\tfrac{1}{3} \text{ yards per second.}$$

Thus the second train is moving 4 yards per second the faster.

(**2**). *Find in feet per second the speed of the earth round the sun, assuming it to describe in 365 days a circle of 92000000 miles radius.*

Taking the value of π as 22/7 the circumference of the earth's orbit is

$$2 \times 22 \times 92000000/7 \text{ miles},$$

or

$$2 \times 22 \times 1760 \times 3 \times 92000000/7 \text{ feet}.$$

Again, 365 days contain $365 \times 24 \times 3600$ seconds.

Hence in $365 \times 24 \times 3600$ seconds the earth moves

$$\frac{2 \times 22 \times 1760 \times 3 \times 92000000}{7} \text{ feet};$$

$$\therefore \text{ the speed is } \frac{2 \times 22 \times 1760 \times 3 \times 92000000}{7 \times 365 \times 24 \times 3600} \text{ feet per second.}$$

And this reduces to 97691 feet per second.

(**3**). *Find the average velocity of a train which takes 7 minutes to traverse the first three miles after leaving a station, then moves for half an hour at the rate of 40 miles an hour, and finally comes to rest, taking 5 minutes to traverse the last 2 miles.*

The whole distance travelled is $3+20+2$ or 25 miles. The time taken is $7+30+5$ or 42 minutes. Thus the average speed is 25/42 or about ·595 mile per minute.

27. Graphical representation of Velocity.

Proposition 3. *To shew that a velocity can be represented by a straight line.*

To determine the value of a velocity we require to know its amount and its direction; these two quantities can be represented by a straight line containing as many units of length as the velocity contains units of velocity and drawn in the direction of motion.

Thus velocities may be completely represented by straight lines.

28. The Composition of Displacements. Consider a man in a railway carriage. Let him move diagonally across the carriage from A to B, Fig. 13. Then if the carriage is at rest his displacement is AB, but suppose the carriage to be in motion and let AA' represent the displacement of any point of the

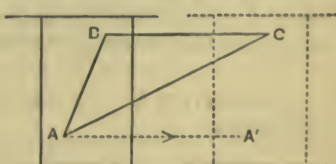


Fig. 13.

carriage during the interval in which the man has moved from A to B. Draw BC equal and parallel to AA'; then relatively to the carriage the man moves from A to B, but relatively to the rails B has moved from B to C, thus the man is at C; his actual displacement[1] is AC, and it is made up of the displacement AB relative to the carriage and BC relatively to the lines.

In this case AC is said to be the *Resultant* of the two displacements AB and AA', and these displacements are spoken of as the *Components* of AC.

Moreover we may continue this process. The rails are not at rest, they are in motion round the axis of the earth.

Let AA'', Fig. 14, represent the displacement of any point on the rails. Draw CD equal and parallel to AA''; then both man, carriage and rails have been displaced through a distance represented by CD; the man therefore will be at D. His displacement is AD, and this is the resultant of AB, BC and CD while these displacements are the components of AD.

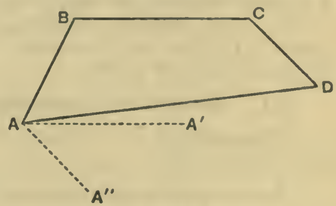


Fig. 14.

DEFINITION. *The single displacement which is equivalent to two or more displacements impressed on a particle is called the* Resultant *of those displacements.*

Each of a number of individual displacements, the combined effect of which is equivalent to a single displacement, is spoken of as a Component of that single displacement.

PROPOSITION 4. *To find the resultant of a number of displacements.*

[1] This does not at all imply that the man has moved along the straight line AC, but merely that he was at A and is at C.

Consider in the first place two displacements OA and OA', Fig. 15. Draw AB equal and parallel to OA'. In consequence of the displacement OA alone the particle would be at A, in consequence of the second displacement OA' the point A is brought to B. Thus the particle is brought to B and its resultant displacement is OB.

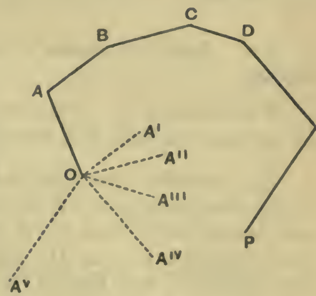


Fig. 15.

Now let there be three displacements OA, OA', OA'', construct the figure as above and draw BC equal and parallel to OA''. In consequence of the displacement OA'', B is brought to C, thus the particle is at C and its displacement is OC.

The general rule, therefore, is obvious. From any point O draw OA to represent the first displacement, from A, the extremity of OA, draw AB to represent the second, from B draw BC to represent the third, and so on. Thus if P be the last point thus found OP is the Resultant displacement.

We notice that the various displacements and their resultant form a closed polygon; if it should happen that the point P should coincide with O it is clear that the resultant displacement is zero; the particle will remain at rest. Thus if a series of displacements can be represented by the sides of a closed polygon taken in order the particle remains at rest. Moreover it is immaterial in what order the displacements are made; we can prove this graphically by drawing the figure in various ways, starting with OA' or OA'' instead of OA, then drawing from A', $A'B'$ to represent OA, and so on; it will be found that we always arrive in the end at the same point P.

In general then, if the several displacements be represented by all but one of the sides of a polygon taken in order their resultant is represented by that side taken in the opposite

direction. This proposition is called the polygon of displacements.

29. Special cases of the composition of displacements.

PROPOSITION 5. *When the component displacements are all in the same straight line the resultant is their algebraical sum.*

For consider two such displacements. Draw OA, Fig. 16, to represent the first, AB to represent the second; then AB is in the same straight line as OA and if OA and AB are drawn in the same direction, Fig. 16, then $OB = OA + AB$.

Fig. 16.

While if OA and AB are drawn in opposite directions, Fig. 17, then

$$OB = OA - AB.$$

Fig. 17.

In either case OB is the *algebraical* sum of OA and AB.

The proposition may clearly be extended to three or more displacements.

30. The Parallelogram of Displacements.

PROPOSITION 6. *If two displacements represented in direction and magnitude by two straight lines* OA, OB *meeting at a point be impressed on a particle, the resultant is* OC, *the diagonal through* O *of the parallelogram which has* OA, OB *for adjacent sides.*

For from A draw AC, Fig. 18, equal and parallel to OB. Join OC and BC. Then $OACB$ is a parallelogram and OC is the diagonal through O.

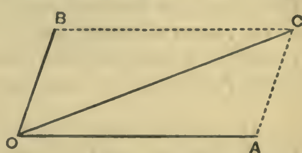


Fig. 18.

In consequence of the displacement OA the particle is moved to A; in consequence of the displacement OB, A is moved to C. Thus OC is the resultant

displacement and it is the diagonal through O of the parallelogram $AOBC$.

This proposition may be put into a slightly different form, thus:

If two sides OA, AC of a triangle OAC, Fig. 19, represent displacements impressed on a particle, then the third side OC represents the resultant displacement. In this form it is known as the Triangle of Displacements.

Fig. 19.

PROPOSITION 7. *To find an expression for the resultant of two displacements at right angles.*

Let OA, OB, Fig. 20, represent two displacements, P, Q respectively, at right angles to each other. Complete the rectangle $AOBC$. Let R be the resultant of P and Q, then R is represented by OC.

Since the angle OAC is a right angle we have

$$OC^2 = OA^2 + AC^2$$

$$= OA^2 + OB^2,$$

$$\therefore \quad R^2 = P^2 + Q^2.$$

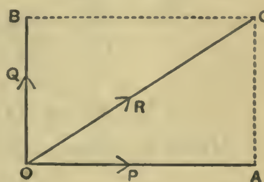


Fig. 20.

Hence $$R = \sqrt{\{P^2 + Q^2\}}.$$

*PROPOSITION 8. *To find an expression for the resultant of two displacements inclined to each other at any angle.*

Let OA, OB represent respectively two displacements P, Q inclined to each other at an angle γ.

Complete the parallelogram $AOBC$. Then OC represents R the resultant of P and Q. Draw CD perpendicular to OA meeting OA produced, Fig. 20 (a), or OA, Fig. 20 (b), in D. Then $AOB = \gamma$; in Fig. 20 (a) the angle γ is less than a right angle; in Fig. 20 (b) it is greater.

Now in Fig. 20 (a),

$$OD = OA + AD = OA + AC\cos DAC$$
$$= OA + OB\cos AOB = P + Q\cos\gamma,$$
$$CD = AC\sin CAD = OB\sin\gamma = Q\sin\gamma.$$

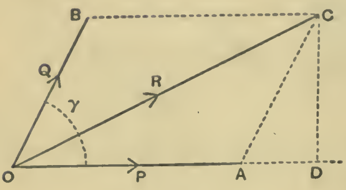


Fig. 20 (a).

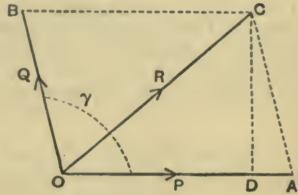


Fig. 20 (b).

In Fig. 20 (b),

$$OD = OA - AD = OA - AC\cos DAC$$
$$= OA - OB\cos(180 - \gamma) = OA + OB\cos\gamma$$
$$= P + Q\cos\gamma,$$
$$CD = AC\sin CAD = OB\sin(180 - \gamma) = Q\sin\gamma.$$

Hence in either case we have

$$R^2 = OC^2 = OD^2 + DC^2$$
$$= (P + Q\cos\gamma)^2 + Q^2\sin^2\gamma$$
$$= P^2 + Q^2 + 2PQ\cos\gamma;$$
$$\therefore\quad R = \sqrt{\{P^2 + Q^2 + 2PQ\cos\gamma\}}.$$

There are many special cases of this last proposition which can be solved by Geometry without reference to Trigonometry. Thus, suppose the angle between the two displacements to be 45°.

Hence, constructing Fig. 21 as above, we have

$$AD^2 + CD^2 = AC^2 = Q^2.$$

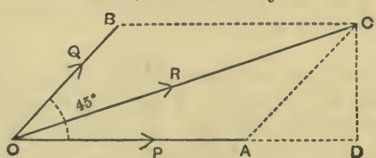


Fig. 21.

Also $$AD = DC;$$

$$\therefore\ AD = DC = \frac{Q}{\sqrt{2}}.$$

And $$OD = OA + AD = P + \frac{Q}{\sqrt{2}}.$$

Hence $$R^2 = OC^2 = OD^2 + DC^2$$
$$= \left(P + \frac{Q}{\sqrt{2}}\right)^2 + \frac{Q^2}{2}$$
$$= P^2 + Q^2 + PQ\sqrt{2}.$$

Or again, if $\gamma = 60°$, we have, Fig. 22,

$$AD = \tfrac{1}{2}AC = \tfrac{1}{2}Q,\quad CD = \frac{Q\sqrt{3}}{2}.$$

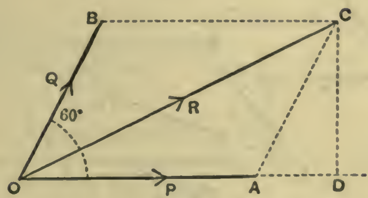


Fig. 22.

$$R^2 = \left(P + \frac{Q}{2}\right)^2 + \frac{3Q^2}{4}$$
$$= P^2 + Q^2 + PQ.$$

These are both given by the general formula by putting $\gamma = 45°$, $\cos\gamma = \frac{1}{\sqrt{2}}$ and $\gamma = 60°$, $\cos\gamma = \frac{1}{2}$ respectively.

If the two displacements be equal the resultant bisects the angle between them; for, Fig. 23, if

$$OA = AC,$$

then $$\angle AOC = \angle ACO$$
$$= \angle BOC.$$

Join AB, cutting OC in D, then AB bisects OC at right angles. And

$$R = OC = 2OD = 2OA \cos AOC$$
$$= 2P \cos\tfrac{1}{2}\gamma.$$

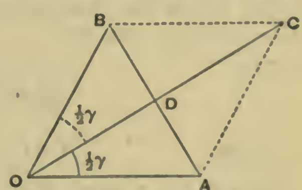


Fig. 23.

31. The Resolution of Displacements.

Just as we can combine or compound two or more displacements and find their resultant, so conversely we can resolve a single displacement into a number of others which are equivalent to it; these are called its components.

PROPOSITION 9. *To find, by a graphical construction, the components of a displacement in any two given directions.*

Let OC, Fig. 24, be the given displacement, and LM, LN the two given directions. Through O draw OA parallel to LM and through C draw AC parallel to LN. These two displace-

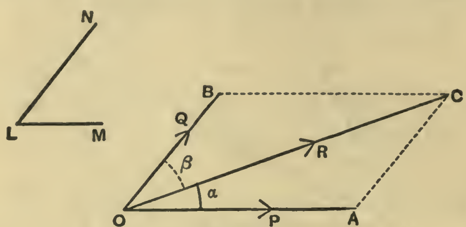


Fig. 24.

ments OA, AC have OC for their resultant, hence OA, AC are components of OC and they are parallel respectively to LM and LN, that is they are drawn in the given directions.

*PROPOSITION 10. *To find an expression for the components of a displacement in two given directions.*

Let OC, Fig. 24, represent R the given displacement, and let OA, AC be the components in directions making angles α, β, respectively with OC.

Then $$AOC = \alpha,$$
$$BOC = ACO = \beta.$$

Hence $$OAC = 180 - (\alpha + \beta).$$

Now in the triangle OAC the sides are proportional to the sines of the opposite angles.

Hence $$\frac{OC}{\sin OAC} = \frac{OA}{\sin ACO} = \frac{AC}{\sin AOC},$$

$$\therefore \quad \frac{R}{\sin(\alpha+\beta)} = \frac{P}{\sin\beta} = \frac{Q}{\sin\alpha}.$$

Moreover from the figure $\alpha+\beta=\gamma$.

Hence $$P = R\frac{\sin\beta}{\sin\gamma}$$

$$Q = R\frac{\sin\alpha}{\sin\gamma}.$$

PROPOSITION 11. *To find the components of a displacement in two directions at right angles.*

Let OC, Fig. 25, be the displacement R, OA, OB two directions at right angles in which the components are required.

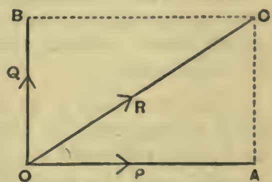


Fig. 25.

Let $AOC = \alpha$.

Draw CA, CB perpendicular to the two directions respectively. Then OA, OB represent the components P, Q.

Also $\dfrac{OA}{OC} = \cos AOC = \cos\alpha$,

$$\therefore \quad OA = OC\cos\alpha.$$

Hence $$P = R\cos\alpha.$$

Again $\dfrac{OB}{OC} = \cos BOC = \sin AOC = \sin\alpha$,

$\therefore \; OB = OC\sin\alpha.$

Hence $Q = R\sin\alpha.$

If we put $BOC = \beta$ we have clearly

$$OB = OC\cos\beta$$

$$Q = R\cos\beta.$$

And in this case $\alpha+\beta = 90^\circ$.

Thus, when a displacement is resolved into two others mutually at right angles, the component in each direction is found by multiplying the original displacement by the cosine of the angle between it and the direction of the component.

It must be remembered that this result is only true when the two components are at right angles.

Thus, let OA, OB (Fig. 26) be two components of OC at right angles. Draw OB' making an angle γ with OA and through C draw CA' parallel to OB'. If now OC be resolved into two displacements in directions OA and OB' inclined at an angle γ, the component in the direction OA is no longer OA but OA'.

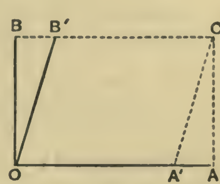


Fig. 26.

A displacement represented by OA is $R\cos\alpha$, where α is the angle between OC and OA; that represented by OA' has not this value.

32. The Composition of Velocities. In the previous propositions we have considered the composition of displacements without reference to the time taken to produce the displacement.

Now a velocity is measured by the displacement produced in the unit of time; if therefore the various lines of the figures represent displacements per second, they represent velocities and the propositions are therefore true of velocities as well as of displacements.

Thus if a point has a number of velocities communicated to it simultaneously, it will move in the direction of the resultant velocity as given by a construction similar to that of § 28 (the polygon of displacements), and its speed will be measured by the length of the line representing that resultant velocity.

It is immaterial to the result how the particle comes to possess these various velocities; thus, a man moving across a railway carriage has his own velocity relative to the carriage, this is superposed on the velocity of the carriage along the rails, this again

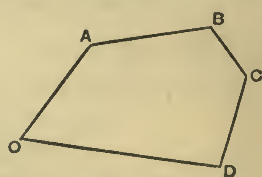


Fig. 27.

on the velocity of the lines about the axis of the Earth, and this on the velocity of the Earth about the Sun; to find the resultant draw OA, Fig. 27, to represent the velocity of the man across the carriage, AB to represent the velocity of the carriage, BC that of the rails round the Earth's axis, and CD that of the Earth's centre round the Sun. Join OD, then OD is the actual velocity of the man relative to the Sun.

Or again, we may suppose the particle to be set in motion by a number of blows delivered simultaneously, one of which would give it a velocity OA, Fig. 28, a second would give a velocity OA', a third OA'', and so on, the resultant is found in the same manner; thus, from O draw OA to represent the first velocity, from A draw AB equal and parallel to OA' to represent the second, from B draw BC equal and parallel to OA'' to represent the third, and so on; then if P is the extremity of the last line thus drawn OP is the resultant velocity.

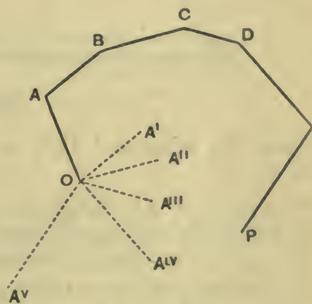


Fig. 28.

Experiment 7. *To illustrate the composition of velocities.*

As an illustration of the composition of velocities, consider the motion of a marble which is rolling with uniform speed u along a tube AB, Fig. 29, and suppose each point of the tube to be moving with uniform speed v parallel to AC. The two motions will be independent, the marble will move relatively to the tube as though the tube were at rest, while the tube moves as though the marble were not present. In AC take AL_1 equal to v, draw L_1M_1 parallel to AB, and make L_1P_1 equal to u. At the end of one second the tube will be in the position L_1M_1, the end A having come to L_1.

Since in one second the marble has moved a distance u

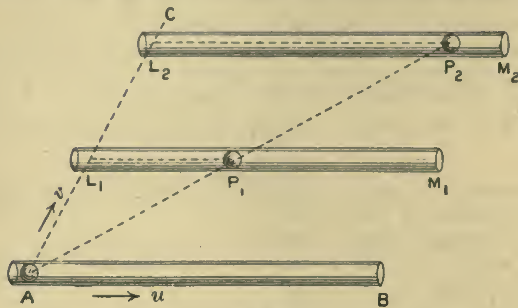


Fig. 29.

along the tube, and P_1L_1 is equal to u, the marble will be at P_1.

Again, if AL_2 is equal to $2v$, then L_2M_2 drawn parallel to AB is the position of the tube after 2 seconds, and if L_2P_2 is equal to $2u$, then P_2 is the position of the marble. Thus the position of the marble at any time can be determined, and it will be seen on making the construction that all the points thus found lie on the line AP_1 or that line produced, thus the marble moves in a straight line AP_1P_2.

Again, AP_1 is the distance traversed in one second, AP_2 the distance traversed in two seconds. Now, from the figure, AP_2 is equal to twice AP_1. Thus the distance traversed in two seconds is equal to twice that traversed in one second; continuing thus we see that the distance traversed is proportional to the time of traversing it, and hence the velocity is uniform. Again, AP_1 is the distance traversed in one second, it is therefore the resultant velocity, and AP_1 is the diagonal of a parallelogram whose sides are u and v. Thus the parallelogram of velocities is verified.

This example illustrates the method adopted in the formal proof of the proposition which is given below in a form slightly modified from the proof of the parallelogram of displacements in § 30.

33. The Parallelogram of Velocities.

PROPOSITION 12. *If a particle possess simultaneously two velocities represented by two adjacent sides of a parallelogram, these are equivalent to a single resultant velocity represented by the diagonal of the parallelogram passing through their point of intersection.*

Let OA, OB, Fig. 30, represent the two velocities u, v respectively. Complete the parallelogram $AOBC$, and draw the diagonal OC. Then OC shall represent the resultant velocity.

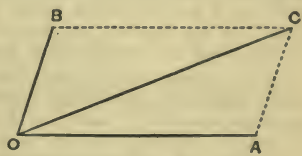


Fig. 30.

(i) Let the two velocities be uniform. Then OA, OB represent the displacements of the point in one second due to the two velocities separately. Now we may consider the motion to be made up of a displacement with velocity u along OA, and a displacement with velocity v of the line OA parallel to itself.

Owing to the first, at the end of one second, the particle would be at A, but owing to the motion of the line OA the point A at the end of one second will have come to C. Thus the particle will be at C.

Again the component velocities are uniform, i.e. the same in magnitude and direction at each instant, hence their resultant must be a uniform velocity; hence the particle has moved in one second from O to C with uniform velocity. Hence the straight line OC represents the resultant velocity.

(ii) When the two velocities are not uniform the proof given in (i) still applies, for a variable velocity can be measured by the distance which would be traversed in one second if during that second the velocity remained constant. Thus OA, OB represent the distances which would be traversed in one second by the particle moving with velocities u, v respectively, if during the second those velocities remained constant. Hence OC represents the distance which would be traversed in one second by the particle when moving with the resultant velocity, if

during the second the resultant velocity remained constant. Hence OC represents the resultant velocity.

Thus whether the component velocities be uniform or variable the diagonal OC still represents their resultant Hence the parallelogram of velocities is true.

There is however an important distinction to be observed between the two cases. Let us suppose that O in Fig. 30 represents the original position of the particle, then *if the velocities be uniform* the position of the particle at the end of one second if it possessed the velocity u only would be A and its actual position is C. OC represents not only the velocity of the particle but also its path; it has moved with uniform velocity along the line OC from O to C.

If the velocities be variable, then OA and OB do not represent the *actual* displacements of the particle in one second due to the two velocities respectively, and therefore OC is not the *actual* displacement due to the resultant velocity. The particle at the end of a second is not at C. The line OC represents the resultant velocity but *not* the path described.

34. Composition and Resolution of Velocities.

The various propositions proved for displacements in §§ 28, 30 may now be extended to velocities. Thus we have the Triangle of Velocities (§ 30). *If two velocities be represented by two sides of a triangle* taken in order their resultant is represented by the third side taken in the reverse direction.

Hence if OAC, Fig. 31, be a triangle and if OA, AC, represent two velocities possessed simultaneously by a particle, then OC represents the resultant velocity.

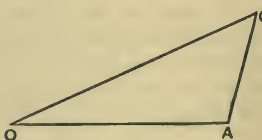


Fig. 31.

Or, putting the same result in another form. If a particle possess velocities represented in direction and magnitude by the three sides of a triangle taken in order it remains at rest.

Resultant of two velocities at right angles, § 30.

If u, v represent the two velocities and U the resultant, then

$$U = (u^2 + v^2)^{\frac{1}{2}}.$$

Resultant of two velocities inclined at an angle γ, § 30.

If u, v be the two velocities, U the resultant,

$$U = \{u^2 + v^2 + 2uv\cos\gamma\}^{\frac{1}{2}}.$$

Components at right angles of a velocity U, § 31.

Let u, v be the two components at right angles, and let u make an angle α with U, then v makes an angle $90° - \alpha$ with U, and we have

$$u = U\cos\alpha$$
$$v = U\cos(90° - \alpha) = U\sin\alpha,$$

u and v are spoken of as the *resolved parts* of the two velocities.

Components in any two directions of a velocity U, § 31.

Let u, v the two components make angles α, β, respectively with U.

$$u = U\frac{\sin\beta}{\sin(\alpha+\beta)},$$

$$v = U\frac{\sin\alpha}{\sin(\alpha+\beta)}.$$

The proofs of these various propositions are identical with those given in the corresponding sections quoted, the word velocity being substituted for displacement. It is left as an exercise to the student to write them out in this form.

Examples. (1). *Find the resultant of velocities of* 2 *to the North*, 3 *to the East*, 3 *to the South, and* 4 *to the West.*

Draw a vertical line OA, Fig. 32 (a), upwards 2 cm. in length, draw BA horizontal to the right 3 cm. in length, BC vertical downwards 3 cm. in length; CD horizontal to the left 4 cm. in length. Join OD, it is the resultant required.

Also if OL be drawn perpendicular on CD it is clear that OL is 1 cm. and LD is 1 cm.

Hence $OD = \sqrt{2}$ cm.

Thus the resultant velocity is $\sqrt{2}$ to the Southwest.

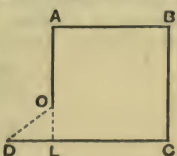


Fig. 32 (a).

Aliter. Velocities of 2 north and 3 south have clearly a resultant of 1 south, velocities of 3 east and 4 west have a resultant 1 west; the resultant of 1 south and 1 west is $\sqrt{2}$ south-west.

(**2**). *A boat is rowed across a river $\frac{1}{2}$ a mile wide with a velocity of 3 miles per hour, and the stream carries it down with a velocity of 4 miles per hour. Find its actual velocity and the distance parallel to the bank between the starting point and the point at which it arrives.*

The velocity of the boat is the resultant of two at right angles of 3 miles an hour and 4 miles an hour respectively, denoting it by U we have

$$U^2 = 3^2 + 4^2 = 25;$$

$$\therefore\ U = 5 \text{ miles per hour.}$$

The time taken to cross the river is independent of the motion downwards. Thus, since the river is $\frac{1}{2}$ a mile wide and the velocity at right angles to the stream is 3 miles an hour, a distance of $\frac{1}{2}$ a mile is traversed in $\frac{1}{6}$ of an hour.

Thus the time of crossing is 10 minutes.

But the stream moves at the rate of 4 miles an hour, thus in $\frac{1}{6}$ of an hour the boat is carried $\frac{2}{3}$ of a mile down.

Thus the distance parallel to the bank between the points is $\frac{2}{3}$rds of a mile.

(**3**). *Find the resultant of two velocities of* 50 *cm. per second and* 100 *cm. per second inclined at an angle of* 60°.

Substituting in the formula

$$U^2 = u^2 + v^2 + 2uv\cos\gamma,$$

we have

$$U^2 = 50^2(1 + 4 + 2 \times 2 \times \tfrac{1}{2})$$
$$= 50^2 \times 7;$$
$$\therefore\ U = 50\sqrt{7} \text{ cm. per second.}$$

This might be solved as in § 30 without quoting the Trigonometrical formula.

(**4**). *A particle has a velocity of* 10 *cm. per second in a north-west direction, find its components to the north and to the west.*

Draw OC, Fig. 32, 10 cm. long to represent the given velocity. From O and C draw OA and CA each at 45° to OC meeting at A, then OA and AC represent the two components. Also from the figure,

$$AO = AC,$$
$$OC^2 = AC^2 + AO^2,$$
$$\therefore\ AC = \frac{10}{\sqrt{2}} = AO.$$

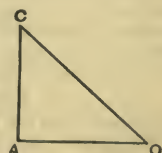


Fig. 32.

Thus each of the components is $\frac{10}{\sqrt{2}}$ cm. per second.

(5). *A particle has a velocity of* 15 *cm. per second which is resolved into two components at right angles. The magnitude of one component is* 9 *cm. per second, find that of the other.*

If u be the other component we have

$$u^2 + 9^2 = 15^2.$$

$$\therefore\ u^2 = 15^2 - 9^2 = (15+9)(15-9) = 24 \times 6.$$

Hence $u = 12$ cm. per second.

(6). *Find an expression for the resultant of a number of velocities* u_1, u_2, u_3, etc. *making angles* a_1, a_2, a_3, etc. *with a fixed line.*

Let U be the resultant and let it make an angle θ with the line.

Then the resolved parts of the resultant in any two directions at right angles must be equal to the resolved parts of the components in these two directions.

Hence resolving along and perpendicular to the fixed line

$$U \cos\theta = u_1 \cos a_1 + u_2 \cos a_2 + u_3 \cos a_3 + \dots$$
$$= \Sigma\{u \cos a\} \quad\dots\dots(1),$$

where Σ is written for abbreviation and means *the sum of a number of terms such as*

$$U \sin\theta = u_1 \sin a_1 + u_2 \sin a_2 + \dots\dots$$
$$= \Sigma\{u \sin a\} \quad\dots\dots(2).$$

Hence squaring and adding, since $\sin^2\theta + \cos^2\theta = 1$,

$$U^2 = [\Sigma\{u \sin a\}]^2 + [\Sigma\{u \cos a\}]^2$$

$$\tan\theta = \frac{\Sigma\{u \sin a\}}{\Sigma\{u \cos a\}},$$

and from these equations the resultant velocity U and its direction θ can be found.

(7). *The resultant of two velocities of* 3 *cm. per sec. and* 5 *cm. per sec. respectively is a velocity of* 7 *cm. per sec. Find the angle between the two.*

Let γ be the angle required, then if u, v be two velocities inclined at an angle γ which have a resultant U we know that

$$U^2 = u^2 + v^2 + 2uv \cos\gamma.$$

Hence
$$7^2 = 5^2 + 3^2 + 2 \times 5 \times 3 \cos\gamma;$$

$$\therefore\ 30 \cos\gamma = 7^2 - 5^2 - 3^2$$
$$= 49 - 25 - 9 = 15,$$
$$\therefore\ \cos\gamma = \tfrac{1}{2},$$
$$\therefore\ \gamma = 60^\circ.$$

Hence the angle between the component velocities is 60°.

35. Experiments on the Parallelogram Law.

The parallelogram law for the composition of displacements and velocities can be illustrated by means of the apparatus shewn in Fig. 33 and described in the following experiment.

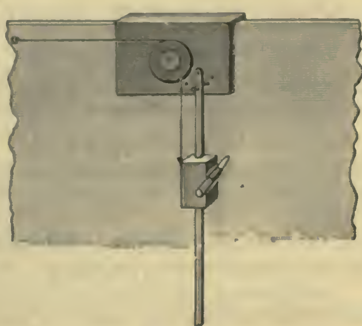


Fig. 33.

EXPERIMENT 8. *To verify the parallelogram law for the composition of velocities.*

A block of wood is made with a groove so as to slip along the horizontal edge of a drawing-board, held with its plane vertical; to this wooden block is fitted a pulley round which a string passes, one end of the string is fastened to the drawing-board and the other to the pulley; as the block is moved along the edge of the board the pulley is rotated about an axis at right angles to the board and the string is unwound. Rigidly attached to this pulley are one or more pulleys of different diameters, which revolve round the same axis as the first; one end of a string is fixed to one of these pulleys, and to the other end is attached a weight sliding on a bar which hangs vertically down from the block or can be fixed at any required angle to the horizon; the weight is thus raised or lowered as the pulley rotates and carries a clip to which a piece of chalk can be attached. If now the block be moved along the edge of the board the weight will be moved horizontally with a certain velocity, that of the block; but owing to the motion of the block the pulleys are rotated

and the string to which the weight is attached is wound up; and so motion along the bar is imparted to the weight; the actual displacement of the weight will be the resultant of these two displacements.

Perform the experiment as follows: fix the bar so as to be vertical; mark the point on the bar at which the weight is, slide the block along the edge of the board through a measured distance, and trace by means of the chalk attached to the weight the path of its motion, it will be found to be a straight line. Measure the distance along the bar through which the weight has moved and draw from the point at which the weight starts two straight lines, one horizontal and equal to the distance moved by the block, the other parallel to the bar and equal to the distance traversed by the weight along it. It will be found that the line marked by the chalk is the diagonal of the parallelogram of which the two lines are sides. Now these two lines represent the component displacements of the weight, and we see that the diagonal represents the resultant. Hence the parallelogram law is verified. Repeat the experiment for other positions of the bar carrying the weight, that is for other angles between the component velocities. The ratio of the displacements in the two directions depends on the diameters of the pulleys used and can be varied by using different sized pulleys.

*36. Relative Velocity.

It has already been pointed out that all motion with which we are concerned is relative motion, and we have seen that a particle A is in motion relative to a second particle B, when the length or direction of the line AB varies. It is often desirable to determine the motion of one particle relative to a second which is itself in motion. Now it is clear that the relative motion of two particles is not altered by superposing on both the same velocity; for example, the relative motion of two flies crawling on the window of a railway carriage is the same, whether the carriage be at rest or in motion.

We can apply this then to find the motion of A relative to B thus. Superpose on the motions of A and B a velocity equal

and opposite to that of B. The relative motion is unaltered, the particle B is reduced to rest while A moves with a velocity which is the resultant of its own velocity and the reversed velocity of B. This resultant motion is now the motion of A relative to B.

Example. *The paths of two ships intersect at right angles, one ship, moving with a velocity of* 15 *miles an hour, is* 15 *miles from the point of intersection, the other, moving with a velocity of* 20 *miles per hour, is* 10 *miles from this point; find the least distance between the ships.*

Let O, Fig. 34, be the point of intersection of the paths, A the position of the first ship 15 miles from O, B that of the second 10 miles from O. Bisect OA in C and join BC, from A draw AD perpendicular to BC produced.

Then $BO=10$ miles,

$OC=CA=7{\cdot}5$ miles,

$BC=12{\cdot}5$ miles,

for $BC^2=BO^2+OC^2$.

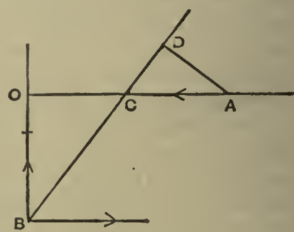


Fig. 34.

If we take BO to represent a velocity of 20 miles an hour, OC will represent one of 15 miles per hour in a direction opposite to that in which A is moving. If then we superpose on the ships a velocity represented by OC, the ship A will be reduced to rest while B will move in the direction BC with a velocity represented on the same scale by BC; this is a velocity of 25 miles an hour. Thus the relative motion of the two ships is represented by a velocity of 25 miles an hour in the direction BC.

The two ships will be nearest apart when B has arrived at D, and this least distance is given by AD.

Now
$$\frac{AD}{AC}=\frac{OB}{BC};$$
$$\therefore AD=\frac{AC \,.\, OB}{BC}=\frac{7{\cdot}5\times 10}{12{\cdot}5}$$
$$=6 \text{ miles}.$$

Thus the least distance apart of the ships will be 6 miles.

Moreover
$$\frac{CD}{AD}=\frac{CO}{OB};$$
whence
$$CD=4{\cdot}5 \text{ miles},$$
$$\therefore BD=17 \text{ miles}.$$

Hence since the relative velocity of B along BD is 25 miles per hour, the ship B will reach D in $\frac{17}{25}$ hour from the time at which it was at B, and at this time the two ships will be the closest together.

*37. Angular Velocity.

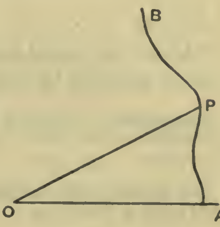


Fig. 35.

Let P, Fig. 35, be a point which is moving along a plane curve APB, and let O be any fixed point in the plane of the curve and OA a fixed line through O. As P moves the angle POA varies; the rate of change of this angle is called the angular velocity of the point P about O, and is measured in general by the ratio of the change in the angle to the interval of time during which that change has occurred when that interval is made sufficiently small.

When the angular velocity is uniform it is measured by the ratio of the angle θ, described in the interval of time t seconds, to the time, so that in this case if ω be the uniform angular velocity we have

$$\omega = \frac{\theta}{t},$$

$$\theta = \omega t.$$

***38. Motion with uniform speed in a circle.** If the curve described be a circle, with O, Fig. 36, as centre, we can find a relation between the uniform angular velocity about O and v the uniform speed of the particle in the circle.

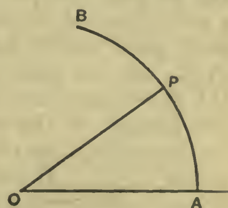


Fig. 36.

For if s be the arc described in time t measured from A we have, since the speed is constant,

$$s = vt.$$

But if a be the radius of the circle, and θ the circular measure of the angle AOP, then

$$\theta = \frac{s}{a}.$$

But $$\theta = \omega t,$$

$$\therefore \quad \omega t = \frac{vt}{a}.$$

Hence $$v = a\omega.$$

Hence the speed in the circle is found by multiplying the angular velocity by the radius of the circle.

Example. *Assuming the earth to be a sphere whose radius is $6{\cdot}436 \times 10^6$ metres, find in metres per second the velocity of a point on the equator.*

The earth rotates uniformly through an angle whose circular measure is 2π (44/7) in 24 hours.

$\therefore$ its angular velocity is $\dfrac{44}{7 \times 24 \times 3600}$.

Hence the velocity of a point on the equator is

$$\frac{44 \times 6436 \times 10^3}{7 \times 24 \times 3600},$$

and this reduces to 468 metres per second.

A relation identical with the above holds between the speed, the radius of the circle and the angular velocity about the centre even when the two are not uniform, provided that v and ω stand for the values of the speed and the angular velocity at the same moment of time.

39. Graphical Representation of Space passed over by a particle. Draw a horizontal line OX, Fig. 37, divide it into a number of equal parts in the points N_1, N_2, N_3 etc. and let each such part represent a small interval of time. From each point draw lines P_1N_1, P_2N_2, etc. at right angles to OX to represent the velocity of the particle at the end of the corresponding interval; join the points P_1P_2.... If the intervals be sufficiently small the line joining these points will be a continuous curve. Such a curve is called a velocity curve; it is defined by the property that if a perpendicular PN be drawn from any point on it to meet the time line OX in N then PN is the velocity of the particle at the time represented by ON.

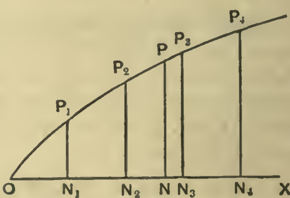


Fig. 37.

Let us now suppose the velocity to be constant; the lines P_1N_1, P_2N_2, etc. will all be of the same length, the velocity curve is a straight line such as PP' parallel to OX, Fig. 38. Let P, P' be two points on the curve and $PN, P'N'$ perpendicular to OX, let t be the time represented by NN' and let v be the constant velocity, s the distance traversed.

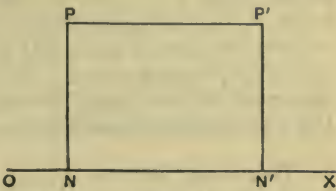


Fig. 38.

Then $$v = PN,$$
$$t = NN'.$$

Now $$s = vt,$$
$$\therefore\ s = PN \times NN' = \text{area } PNN'P'.$$

Thus in this case of uniform velocity *the area between the velocity curve, the line* OX *and two lines perpendicular to the line* OX *represents graphically the space traversed by the particle.*

Some further consideration shews us that this proposition is always true whether the velocity be uniform or variable.

For we have seen that we may approach the case of a *continuously* varying velocity by dividing the time up into a large number of small intervals and supposing the velocity to remain constant during each interval but to change suddenly at the end of every interval.

Draw the velocity curve for such a case supposing for the present the intervals during which the velocity is constant to be *seconds*. It will consist, as shewn in Fig. 39, of the series of horizontal and vertical straight lines

$$P_1R_1P_2R_2P_3R_3 \ldots$$

alternately parallel and perpendicular to OX. During the time N_1N_2

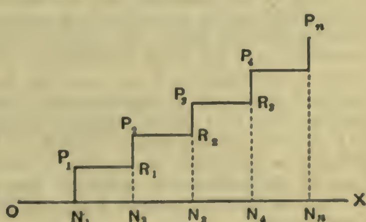


Fig. 39.

the velocity is constant and equal to P_1N_1, the space traversed is the area $P_1N_1N_2R_1$; at the time N_2 the velocity changes to P_2N_2 increasing by P_2R_1; for the next second N_2N_3 the velocity is constant and equal to P_2N_2, the space is represented by $P_2N_2N_3R_2$, and so on. Thus in this case the whole space traversed is the area between the velocity curve, the line OX and the two bounding lines P_1N_1 and P_nN_n.

This result will be equally true if we divide each second into a large number of parts and suppose the velocity to change at the end of each part. Instead of a single step between P_1 and P_2 we obtain a large number of steps; instead of a single parallelogram such as $P_1N_1N_2R_1$ we have a large number; the sum of the areas of these parallelograms is still the space traversed. If now we make the number of parts sufficient each individual step will be indefinitely small, the broken line will merge into the continuous velocity curve, and the sum of the parallelograms will become the area of that curve. Thus the space traversed during a given time is given by the area bounded by the velocity curve, the line OX and two lines perpendicular to OX drawn from points on OX which represent the beginning and the end of the time. Whenever then we can calculate the area of this curve, we can find the space traversed by the particle. In the important case considered in the next chapter the velocity increases uniformly with the time and the curve is a straight line. The area is bounded by straight lines and can therefore be easily calculated.

By drawing a diagram to scale on squared paper and then determining the area by the method given in Experiment 3, we can find the space traversed in many cases in which we are given the relation between the velocity and the time. In solving such a question it is necessary to be careful as to the units in which the lines in the diagram are measured. Suppose, for example, 1 cm. along the time line represents an interval of 1 second, and 1 cm. at right angles to this a velocity of 1 cm. per second; then an area of 1 sq. cm. represents the distance traversed in 1 second by a particle moving with a velocity of 1 cm. per second; that is, it represents a line 1 cm. in length; if however we had taken a length of 1 cm. at right angles to the time line to represent a velocity of v cm. per second, then an area of 1 sq. centimetre would represent a length of v centimetres.

Examples.

(**1**). The velocities at the ends of 1, 2...10 seconds are 5, 7, 9...23 cm. per second, find by a diagram the space traversed in 10 seconds.

(**2**). The velocities at the ends of 1, 2...10 seconds are $1^2, 2^2...10^2$ cm. per second, find by a diagram the space traversed in 10 seconds.

These examples are left for the student to solve with the aid of a ruler and squared paper.

EXAMPLES.

UNIFORM SPEED.

1. Find in feet per second the following velocities:

(1) 10 miles per hour; (2) a quarter of a mile in 44 seconds; (3) 92000000 miles in $8\frac{1}{4}$ minutes; (4) 25000 miles in 24 hours.

2. Find in centimetres per second the velocity of a body which traverses

(1) a cm. in b seconds; (2) a circle of 10 cm. radius in 1 second; (3) 76 cm. in 10 minutes; (4) the perimeter of a square 1 foot in edge in 1 minute.

3. The speed of a steamer is 22 knots, reduce this to cm. per second.

4. A particle has a velocity of 30 miles per hour, how many feet does it traverse (1) in 1 minute, (2) in a day, (3) in a year?

5. A man walks a mile in 10 minutes, a second mile in 12 and a third in 15; he runs a fourth mile in 5 minutes; find his average speed (1) in feet per second, (2) in miles per hour.

6. A and B start to walk towards each other from two places 6 miles apart. A walks twice as fast as B. Where will they meet? The meeting takes place 50 minutes after the start, find the speed of each.

7. A starts along a road at a speed of 3 miles an hour, after 40 minutes B follows at a speed of 5 miles an hour, how far must B go before overtaking A?

8. The velocity of sound is 1100 feet per second, a man in front of a cliff claps his hands and hears an echo after 5 seconds, how far is he from the cliff?

9. A man climbs a hill inclined at 30° to the horizon, if he rises vertically 1000 feet in an hour find his speed in feet per second.

10. The radius of the Earth's Orbit is 92 million miles and the radius of the Earth 4000 miles, compare the velocities of a point on the equator at midday and at midnight.

11. Find the resultants of the following pairs of velocities in directions at right angles to each other; the velocities are all expressed in centimetres per second:

(1) 3 and 4; (2) 6 and 8; (3) 12 and 15; (4) v_1 and v_2, where $v_1+v_2=7$, $v_1-v_2=1$.

12. Find by a graphical construction and by the formulæ the resultants of the following velocities:

(1) 3 and 4 at 60°; (2) 6 and 8 at 45°; (3) 1 and 2 at 30°; (4) 1 and 2 at 60°.

13. A boat is rowed across a river at the rate of 3 miles per hour, the river is flowing at the rate of 4 miles per hour; find the velocity of the boat.

14. A ship is sailing at the rate of 10 miles an hour and a sailor climbs the mast 200 feet high in 30 seconds. Find his velocity relative to the Earth.

15. The paths of two ships steaming North and East respectively, with velocities of 12 and 16 miles per hour, meet. The two ships are each 12 miles distant from the point of intersection. Determine after what time they will be closest together and what that closest distance will be.

16. Two equal velocities have a resultant equal to either, shew that they are inclined to each other at 120°.

17. The resultant of two velocities u and v is equal to u, and its direction is at right angles to that of u. Shew that v is equal to $u\sqrt{2}$.

18. Find by a graphical construction or otherwise the resultant of the following velocities in the directions of the sides of a square taken in order:

(1) 1, 2, 2, 1; (2) 3, 4, 5, 6; (3) 2, 5, 6, 3; (4) 7, 8, 4, 5.

19. Find by a graphical construction or otherwise the resultant of the following velocities in the directions of the sides of an equilateral triangle taken in order:

(1) 3, 3, 3; (2) 4, 5, 6; (3) 5, 8, 10; (4) 6, 9, 12.

20. Find the horizontal and vertical components of the following velocities:

(1) 1000 ft. per second in directions inclined respectively at 30°, 45° and 60° to the horizon.

(2) 25 miles per hour at 60° to the vertical.

21. Resolve a velocity of 1000 feet per second into two equal velocities inclined at 60° to each other.

22. A velocity of 500 feet per second is resolved into two at right angles, one of these is 250 feet per second, find the other.

23. A velocity of 5 miles an hour to the East is changed into one of 5 miles an hour to the North; find the change in velocity.

24. A velocity represented by one side AB of an equilateral triangle ABC becomes changed into one represented by the side AC; find the change in velocity.

25. Find by a graphical construction or otherwise the resultant of the following velocities, which are given in centimetres per second:

(a) 15 to the North, 20 to the East, $20\sqrt{2}$ to the North-west, 35 to the West.

(b) 1, 2, 3, 4, 5, 6 parallel respectively to the sides of a regular hexagon.

26. One of the rectangular components of a velocity of 60 miles per hour is a velocity of 30 miles per hour; find the other component.

27. A body moves during each of 5 consecutive minutes with velocities respectively of 1, 2, 3, 4, 5 feet per second; find the space traversed.

28. The spaces traversed up to the end of 1, 2, 3 and 4 minutes by a body moving with constant velocity during each minute are 2, 8, 18 and 32 feet respectively. Shew on a diagram the velocity during each minute.

29. The velocity of a body starting from rest increases uniformly by 1 foot per second at the end of every second of its motion. Determine by means of a diagram or otherwise the space passed over in t seconds.

30. The components in two directions of a velocity of 30 miles per hour are velocities of 15 and 25 miles per hour, determine their directions.

31. Two velocities u and v have a resultant U which makes an angle α with the direction of u; if u be increased by U while v is unchanged shew that the new resultant makes an angle $\frac{\alpha}{2}$ with the direction of u.

32. Two particles are projected simultaneously with equal velocities from the points A and B, one from A towards B, and the other in a direction at right angles to AB; find how far the former will have travelled towards B when the two particles are nearest to one another.

33. If a point begins to move with velocity u, and at equal intervals of time τ, a velocity v is communicated to it; find the space described in n such intervals.

34. Compare the velocities of two trains, one travelling with a velocity of 50 miles per hour and the other with a velocity of 55 feet per second.

CHAPTER III.

KINEMATICS. ACCELERATION.

40. Change of Velocity. The velocity of a particle may change either in magnitude or in direction or in both these respects.

Let OA represent the velocity of a particle at a given instant; if the velocity remain uniform, OA will continue to represent it; suppose however that the velocity changes and that after an interval it is represented by OB. If the change occur in the magnitude only, the particle will continue to move in the same direction as before.

OAB will be a straight line, and AB, Fig. 40, will represent the velocity which must be added to the original velocity OA to give OB.

Fig. 40.

If the change in velocity take place in direction as well as in magnitude, OA and OB will be inclined to each other, Fig. 41, but AB will still represent the change in the velocity, for by the parallelogram of velocities AB is the velocity which must be compounded with OA to give OB.

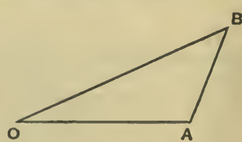


Fig. 41.

Thus AB represents in direction and magnitude the velocity which must be superposed on OA to change it into OB.

If then we represent by two straight lines drawn from a point the initial and final velocities of a particle, the line joining the extremities of these two lines represents the **Change of velocity** of the particle.

Examples. (1). *A particle moving North-east with a velocity of* 1 *foot per second is observed, after a time, to be moving East with a velocity of* $\sqrt{2}$ *feet per second, find the change in velocity.*

Here, Fig. 42,

$$OA = 1, OB = \sqrt{2} \text{ and } AOB = 45^\circ.$$

Draw AC normal to OA meeting OB in C; then $ACB = 45^\circ$ and $AC = AO = 1$.

Hence $OC^2 = 2 = OB^2$, thus C and B coincide, and AB the velocity added is 1 ft. per second in the South-east direction.

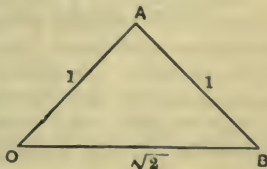


Fig. 42.

(2). *A velocity of* 10 *feet per second is changed into one of* 10 *feet per second inclined at* 60° *to the former, find the change in velocity.*

In this case,

$$OA = OB = 10 \text{ and } AOB = 60,$$

$$\therefore \angle OAB = \angle OBA = \tfrac{1}{2}(180 - 60) = 60^\circ.$$

Thus $\qquad AB = OA = 10.$

The additional velocity is one of 10 feet per second inclined at 60° to OA.

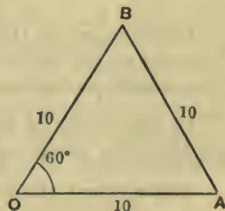


Fig. 43.

41. Acceleration. When the velocity of a particle is variable it is said to have acceleration.

DEFINITION. *The* **Acceleration** *of a particle is its rate of change of velocity.*

Acceleration may be **Uniform** or **Variable.** Uniform acceleration is measured by the ratio of the change of velocity to the interval of time during which that change has occurred, i.e. by the change of velocity in one second; variable accelera-

tion is measured by the same ratio when the interval is sufficiently small, that is, by the change in velocity which would take place in one second if during that second the velocity changed uniformly.

The numerical measure of an acceleration is the number of units of velocity added per second. Now velocity is measured by the number of units of space traversed per second.

When, then, we state that the acceleration of a particle moving with uniform acceleration is a, we mean that in each second an additional velocity of a cm. per second is given to the particle. To define then an acceleration we must know the number of units of space per unit time which are added to the velocity, and further we must remember that this additional velocity is conferred in the unit of time.

Just then as when considering a velocity we speak of so many centimetres per second, so when dealing with acceleration we speak of so many centimetres per second per second.

DEFINITION OF UNIT ACCELERATION. *A particle has* **Unit Acceleration** *when its velocity increases in each second by* 1 *centimetre per second.*

If the units of space or time be changed the numerical measure of a given acceleration is changed also.

The method of calculating these changes is shewn below.

Example. *A particle has an acceleration of* 32·2 *feet per second per second, find its value* (*a*) *in cm. per sec. per sec.*, (*b*) *in yds. per min. per min.*

For (*a*) we have 1 ft. = 30·48 cm.

Now in 1 sec. a vel. of 32·2 ft. per sec. is added,

∴ in 1 sec. a vel. of 32·2 × 30·48 cm. per sec. is added.

∴ the new measure is 32·2 × 30·48 cm. per sec. per sec,

This reduces to 981·5 cm. per sec. per sec.

For (*b*), 1 ft. = ⅓ yd., 1 min. = 60 sec.

In 1 sec. a vel. of 32·2 ft. per sec. is added,

$$\therefore \text{ in 1 sec. a vel. of } \frac{32 \cdot 2}{3} \text{ yds. per sec. is added,}$$

$$\therefore \text{ in 1 sec. a vel. of } \frac{32 \cdot 2 \times 60}{3} \text{ yds. per min. is added,}$$

$$\therefore \text{ in 1 min. a vel. of } \frac{32 \cdot 2 \times 60 \times 60}{3} \text{ yds. per min. is added.}$$

Thus the new measure is $\dfrac{32 \cdot 2 \times 60 \times 60}{3}$ yds. per min. per min.

This reduces to 38640 yds. per min. per min.

It will be noticed that in (*b*) the change in the unit of time comes in twice. The reason for this is clear, the unit of time affects the measure of the velocity and affects also the time during which, when calculating the acceleration, the change in the velocity is to be reckoned.

42. Uniform acceleration in the direction of motion. The change in the velocity of a body may be a change in magnitude, in direction or in both.

For the present we deal only with the case of a body moving in a straight line with uniform acceleration.

The change of velocity will be one of magnitude only, and that change will be a uniform one, the speed will vary but not the direction of motion. The velocity may either increase or decrease; in the first case the acceleration is positive, in the second negative.

PROPOSITION 13. *To determine the velocity of a body moving in a straight line with uniform acceleration in terms of the initial velocity, the acceleration and the time of motion.*

Let the initial velocity be u, the velocity after t seconds v, and the acceleration a.

In 1 second a velocity of a centimetres per second is added and the acceleration is uniform.

Hence in 2″ the velocity added is $2a$,

in 3″ the velocity added is $3a$,

and in t'' the velocity added is at.

Thus at the end of t seconds the velocity is

$$u + at.$$

Hence $$v = u + at.$$

If the velocity decreases with the time, the acceleration is negative and we have

$$v = u - at.$$

The proposition can be put rather differently thus.

The change in velocity in t'' is $v - u$. Therefore the change per second is $(v - u)/t$.

But the acceleration is the change of velocity per second.

Hence $$a = \frac{v - u}{t},$$

$$\therefore\ v - u = at, \text{ or } v = u + at.$$

PROPOSITION 14. *To draw*[1] *the velocity curve for a particle moving with uniform acceleration.*

Draw a horizontal line OX, Fig. 44, to represent time and a vertical line OY to represent velocity. Choose a convenient length to represent the unit of time, and also a convenient length to represent the unit of velocity.

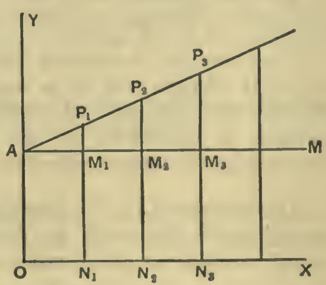


Fig. 44.

Mark off along OY a length OA to represent the initial velocity u.

Through A draw AM parallel to the time line and, commencing from A, divide AM in M_1, M_2, M_3, etc. into equal parts, each of which shall represent 1 second. At M_1, M_2, etc. draw P_1M_1, P_2M_2, P_3M_3, etc. perpendicular to AM, and produce these to meet the time line OX in N_1, N_2, etc. Make P_1M_1 equal to a, P_2M_2 equal to $2a$, P_3M_3 equal to $3a$, etc. Then these

[1] For this and similar purposes squared paper such as is used in Experiment 3 is convenient.

various lines represent the increments of the velocity up to the end of the first, second, third, etc. second, and the lines P_1N_1, P_2N_2, P_3N_3, etc. represent the actual velocity at the end of the first, second, third, etc. second.

Thus the line AP_1P_2 ... represents the velocity curve and the construction shews that it is a straight line.

Examples. **(1)**. *A particle moving from rest with uniform acceleration has a velocity of* 160 *ft. per second after* 5 *seconds, find its acceleration.*

In each second a velocity of 160/5 feet per second is produced.
Hence the acceleration $=\frac{160}{5}=32$ ft. per sec. per sec.

(2). *A particle moving under a negative acceleration of* 32 *feet per second per second is projected with a velocity of* 160 *feet per second. Find when it will come to rest and what will be the velocity after* 10 *seconds.*

In each second a velocity of 32 feet per second is destroyed.

Therefore the initial velocity of 160 feet per second will be destroyed in 160/32 seconds.

Thus the particle is instantaneously at rest after 5 seconds.

The acceleration now produces in each second a velocity in the opposite direction of 32 feet per second. Therefore after 5 seconds more, i.e. at the end of 10 seconds, the velocity will be

$$5\times(-32) \text{ or } -160 \text{ feet per second.}$$

Aliter. Let v be the velocity after t seconds.

Then $$v=160-32\,.\,t.$$

If t_1 represent the time at which the particle is at rest, at which therefore v is zero, we have $O=160-32t_1$;

$$\therefore\ t_1=\tfrac{160}{32}=5 \text{ seconds.}$$

Again after 10 seconds,

$$v=160-32\times10=-160 \text{ ft. per sec.}$$

(3). *Draw the velocity curve in the case of* (2).

Draw the time and velocity lines OX and OY, Fig. 45. In OY take OA to represent a velocity of 160 ft. per second. Draw a line from A parallel to OX and in it take M_1 so that AM_1 may represent 1 second; from M_1 draw M_1P_1 vertically down to represent a velocity of 32 feet per second. Join AP_1 and produce it, AP_1 is the required velocity curve. It meets the line OX in N, where from the figure,

$$ON=5AM_1.$$

Hence ON represents 5 seconds.

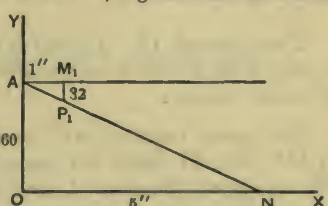


Fig. 45.

43. Acceleration, space traversed and time of motion.

PROPOSITION 15. *To find the space passed over in a given time by a body starting from rest and moving with uniform acceleration.*

The space passed over is given by the area of the velocity curve which, in this case, will be a straight line passing through the point O from which the time and velocity lines are drawn. Let ON, Fig. 46, represent the time t and NP perpendicular to ON the velocity at the end of the time interval.

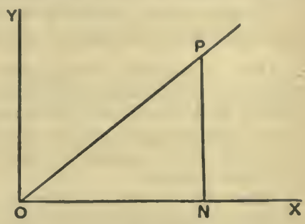


Fig. 46.

Then $$PN = at.$$

Join OP; the velocity curve is the line OP and the space s required is the area of the triangle OPN.

Now the area of a triangle is half the product of the base and the altitude;

$$\therefore\ s = \text{area } OPN = \tfrac{1}{2} PN \, . \, ON$$
$$= \tfrac{1}{2} at \, . \, t = \tfrac{1}{2} at^2.$$

Hence $$s = \tfrac{1}{2} at^2.$$

Thus the space passed over in the first second is $\frac{1}{2} a$ while the velocity at the end of that second is a. *The space traversed is found by multiplying half the acceleration by the square of the time.*

PROPOSITION 16. *To find the space passed over by a particle moving with uniform acceleration when the particle starts with an initial velocity.*

The space required will be the area of the velocity curve. In OY, Fig. 47, take OA equal to the initial velocity u; let ON represent the time t, and NP perpendicular to OX, the velocity after t seconds,

so that $$PN = u + at.$$

Join AP. Then AP will be the velocity curve and the space required is the area $OAPN$.

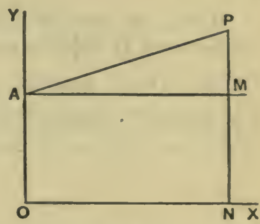


Fig. 47.

Draw AM parallel to OX to meet PN in M.

Then $PM = at$, $MN = OA = u$.

Hence

$$\begin{aligned} s &= \text{area } OAPN \\ &= \text{parallelogram } OAMN + \text{triangle } APM \\ &= OA \times ON + \tfrac{1}{2} PM \times AM, \end{aligned}$$

and $$AM = ON = t.$$

Hence $$\begin{aligned} s &= ut + \tfrac{1}{2} at \,.\, t \\ &= ut + \tfrac{1}{2} at^2. \end{aligned}$$

Thus the space actually traversed is found by adding together the spaces the particle would traverse (1) if it moved with the constant velocity u and (2) if it started from rest with the constant acceleration a.

Proposition 17. *To find the average velocity of a particle moving with uniform acceleration.*

We can put the last formula in a different form thus: let v be the final velocity of the particle, then $v = PN$.

Join AN, then the figure $OAPN$ is made up of the two triangles OAN and PAN; the bases of these triangles are OA and PN and their altitude is ON.

Thus

$$\begin{aligned} s &= \text{area } OAPN = \text{triangle } OAN + \text{triangle } APN \\ &= \tfrac{1}{2} OA \,.\, ON + \tfrac{1}{2} PN \,.\, ON \\ &= \tfrac{1}{2} (OA + PN)\, ON \\ &= \tfrac{1}{2} (u + v)\, t. \end{aligned}$$

Now we know, § 24, that when a particle moves with variable speed the space traversed is found by multiplying the average speed and the time.

In this case, therefore, the average speed so defined is $\frac{1}{2}(u+v)$.

Thus, *the average speed of a particle moving with uniform acceleration is half the sum of the initial and final speeds.*

The above formula may be applied to the case in which a is negative, we have then

$$s = ut - \tfrac{1}{2}at^2$$

$$= \tfrac{1}{2}(u+v)t,$$

for $$v = u - at.$$

In a later chapter (§ 68) experiments will be given by which the student can verify for himself the truth of the formulæ proved in this section. Those who have a difficulty in following the steps of the proof may adopt the experimental proof. The argument just given may be made clearer to some by giving the following algebraical proof which sums up in symbols its results.

Let us suppose the whole time t divided up into a series of n equal small intervals each equal to τ, so that $n\tau = t$.

At the beginning of each interval the velocity will have the values respectively u, $u+a\tau$, $u+2a\tau, \ldots u+\overline{n-1}\,a\tau$, and at the end of each interval it will have the values

$$u+a\tau, \quad u+2a\tau, \quad u+3a\tau, \ldots u+na\tau.$$

The space traversed will be greater than that which would be traversed, if during each interval the particle moved with the velocity which it has at the beginning of the interval, and less than that which would be traversed if during each interval the particle moved with the velocity it has at the end of the interval.

In the first case the space would be s_1, where we have

$$s_1 = u\tau + (u+a\tau)\tau + (u+2a\tau)\tau + \ldots\ldots + (u+\overline{n-1}\,a)\tau;$$

$$\therefore\ s_1 = un\tau + a\tau^2(1+2+\ldots n-1)$$

$$= un\tau + a\tau^2\frac{n(n-1)}{2}.$$

Now $$\tau = \frac{t}{n};$$

$$\therefore\ s_1 = ut + \frac{1}{2}at^2\left(1-\frac{1}{n}\right),$$

and in the second case the space would be s_2, where

$$\begin{aligned} s_2 &= (u+a\tau)\,\tau + (u+2a\tau)\,\tau + \dots\dots + (u+na\tau)\,\tau \\ &= un\tau + a\tau^2(1+2+3+\dots+n) \\ &= ut + \frac{1}{2}at^2\left(1+\frac{1}{n}\right). \end{aligned}$$

But s lies between s_1 and s_2 and these two quantities can both be made as nearly equal as we please to $ut+\frac{1}{2}at^2$ by making n very large, for then $1/n$ vanishes.

Hence $$s = ut + \tfrac{1}{2}at^2.$$

44. Acceleration, Velocity and Space traversed.

PROPOSITION 18. *To find a relation between the velocity, the acceleration, and the space traversed for a particle moving with uniform acceleration.*

Let u be the initial velocity, v the velocity after t seconds during which the particle has traversed a distance s and a the acceleration.

We have proved that

$$v - u = at,$$
$$s = ut + \tfrac{1}{2}at^2.$$

We wish to eliminate t from these equations.

The first gives us

$$t = \frac{v-u}{a}.$$

Hence $$s = \frac{u(v-u)}{a} + \frac{1}{2}a\frac{(v-u)^2}{a^2},$$

$$\begin{aligned} \therefore\ 2as &= 2uv - 2u^2 + v^2 + u^2 - 2uv \\ &= v^2 - u^2. \end{aligned}$$

Hence $$v^2 = u^2 + 2as.$$

If the acceleration be negative we start with

$$v - u = -at,$$
$$s = ut - \tfrac{1}{2}at^2,$$

and find $$v^2 - u^2 = -2as.$$

We can put the proof otherwise, thus

we have $$a=\frac{v-u}{t},$$

$$s=\tfrac{1}{2}(v+u)t.$$

Hence on multiplication,

$$as=\tfrac{1}{2}(v-u)(v+u)=\tfrac{1}{2}(v^2-u^2),$$

or $$v^2=u^2+2as.$$

45. Formulæ connected with uniform acceleration. We have thus proved the following formulæ in which the symbols have the meanings attached to them in the preceding sections.

$$v=u+at \quad \text{.........................(i)},$$

$$s=ut+\tfrac{1}{2}at^2 \quad \text{.........................(ii)},$$

$$v^2=u^2+2as \quad \text{.........................(iii)}.$$

We may also write (ii) as

$$s=\tfrac{1}{2}(v+u)t \quad \text{.....................(iv)}.$$

If the particle start from rest u is zero, and the equations become

$$v=at \quad \text{.........................(i) } a,$$

$$s=\tfrac{1}{2}at^2 \quad \text{.........................(ii) } a,$$

$$v^2=2as \quad \text{.........................(iii) } a,$$

$$s=\tfrac{1}{2}vt \quad \text{.........................(iv) } a.$$

Examples. (1). *A particle starts with a velocity of* 3 *cm. per sec. and an acceleration of* 2 *cm. per sec. per sec., find its velocity after* 10 *sec. and the distance traversed in* 10 *sec.*

Let v be the velocity after 10 seconds, s the distance traversed.

Then $$v=3+2\times 10=23 \text{ cm. per sec.}$$

$$s=3\times 10+\tfrac{1}{2}\,.\,2\,.\,10^2=130 \text{ cm.}$$

(2). *How far must the particle, moving as in Example* (1), *move in order that its velocity may become* 5 *cm. per second?*

To solve this we need equation iii, giving a relation between the velocity and the space.

If the space required be s cm., we have

$$5^2=3^2+2\times 2\times s,$$

$$\therefore\ 4s=5^2-3^2=16;$$

$$s=4 \text{ cm.}$$

(3). *A particle has a velocity of* 20 *cm. per second and an acceleration of* -5 *cm. per sec. per sec., how far will it move before coming to rest?*

If v be the velocity after it has traversed s cm. we have

$$v^2 = 20^2 - 2 \times 5 \times s.$$

If the particle is at rest for a moment, we have v zero, and hence

$$10s = 20^2 = 400;$$
$$\therefore\ s = 40 \text{ cm.}$$

Thus the particle is brought to rest after moving 40 cm.; if the acceleration continue to act it will only remain at rest for an instant, and then commence to retrace its path passing through the starting point with its initial velocity.

(4). *A particle starts with a velocity* u *and an acceleration* $-$a*; shew that it comes to rest after an interval* u/a *seconds and passes through the starting point again after an interval* 2u/a *seconds.*

The velocity after t seconds is $u - at$; when the particle is at rest this is zero,

$$\therefore\ u - at = 0;$$
$$\therefore\ t = \frac{u}{a}.$$

The distance of the particle from the starting point at t'' is

$$ut - \tfrac{1}{2}at^2.$$

When the particle is at the starting point this distance is zero. Then

$$ut - \tfrac{1}{2}at^2 = 0, \quad \therefore\ t = 0;$$

or
$$u - \tfrac{1}{2}at = 0;$$

whence
$$t = \frac{2u}{a}.$$

Thus the particle is at the starting point initially and reaches it again after an interval $2u/a$.

During half this interval the particle is moving from the starting point, during the second half it is moving to it.

(5). *A particle has an initial velocity of* 125 *cm. per sec. and an acceleration of* (*a*) 10 *cm. per sec. per sec.*, (*b*) -10 *cm. per sec. per sec. How long will it take in either case to move over* 420 *cm.?*

We know the initial velocity, the distance traversed and the acceleration and require to find the time.

This is given us by equation ii. Let it be t seconds, then for (*a*),

$$420 = 125t + \tfrac{1}{2}10t^2,$$
$$\therefore\ t^2 + 25t - 84 = 0;$$

solving the quadratic

$$t=\frac{-25\pm\sqrt{625+336}}{2}$$

$$=\frac{-25\pm31}{2}=3 \text{ or } -28.$$

From the solution $t=3$ we see that, 3 seconds after starting, the particle will be at a distance of 420 cm. from the starting point. Now as to the solution $t=-28''$, we infer from this that it is possible to start the particle from a position 420 cm. from the starting point with such a velocity that after 28 seconds it is at the starting point and is moving with a velocity of 125 cm. per second. Let O, Fig. 48, be the original starting point, A a point 420 cm. to the right of O; then if the particle is projected towards A with a velocity of 125 cm. per second it will arrive at A in 3 seconds, this is the first solution. But it is also possible to start the particle from A towards O with such a velocity that it passes through O to B, comes to rest for an instant at B and then returns to O, arriving at O with a velocity of 125 cm. per second, 28 seconds after it has left A; if this be possible then we may say that the particle was at A -28 seconds before leaving O.

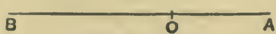


Fig. 48.

And this is clearly possible. In the first case, the particle arrives at O with a velocity of 155 cm. per second, viz. its original velocity of 125 cm. per sec. and the velocity of 30 cm. per sec. generated in 3″ by the acceleration. Suppose now it be projected from A towards O with this velocity. It will arrive at O after 3 seconds and have a velocity of 125 cm. per second; it will then continue to move towards B for 12·5 seconds, in which time the velocity of 125 cm. per sec. will be destroyed. Thus it arrives at B 15·5 secs. after leaving A. It will then return from B to O and will arrive at O after another 12·5 secs. with the velocity of 125 cm. per second. Thus the interval between the start at A and the time at which the particle reaches O with a velocity of 125 cm. per sec. towards A is 15·5 + 12·5 or 28 seconds. Thus under the given conditions of acceleration and velocity the particle might be at A, 420 cm. from O, either 3 seconds after passing O, or 28 seconds before passing O.

(*b*) Taking now the other case in which the acceleration is -10, we have

$$420=125t-\tfrac{1}{2}\,.\,10\,.\,t^2,$$

$$\therefore\; t^2-25t+84=0;$$

$$t=\frac{25\pm\sqrt{(625-336)}}{2}$$

$$=\frac{25\pm17}{2}=4 \text{ or } 21.$$

The reason for the double value of t is clear as before. The particle starts from O and arrives at A after 4 seconds, during which time its

velocity has been reduced to 125 − 40 or 85 cm. per second. It moves on with decreasing velocity until it is brought to rest at B after a further interval 85/10 or 8·5 seconds. Thus it takes 12·5 seconds to reach B from O. It now returns towards O under the acceleration 10 starting from rest at B and after a further interval of 8·5 seconds again passes through A. Thus it reaches A the second time $12·5 + 8·5$ or 21 seconds from the start. The figure for this case is different.

46. Falling Bodies. When a body is allowed to fall to the earth's surface from a point above it, it is found (1) that the acceleration is uniform[1], (2) that the acceleration is the same for all bodies.

The Experiments on which these statements are based will be described later. See §§ 65, 129.

This uniform acceleration of all bodies when falling from a given point is spoken of as the acceleration of gravity, or better, the acceleration due to gravity. It is usually denoted by the symbol g.

Again, Experiment shews us that the acceleration of a falling body differs slightly at different places on the earth; it is greatest at the poles and least at the equator. A body falls from a given height more rapidly at the pole than at the equator.

The value at the pole is 983·11 cm. per sec. per sec. and at the equator 978·10 cm. per sec. per sec.

At Greenwich the value is 981·17 cm. per sec. per sec.

Since 1 foot contains 30·48 cm. the value of g at Greenwich is 981·17/30·48 or 32·191 feet per sec. per sec.[2].

Hence we see that the formulæ of Section 45 are all applicable to the case of a falling body.

[1] This statement is only true for distances above the surface which are small compared with the radius of the Earth (4000 miles). It may be applied therefore without error to the experiments described in this book.

[2] See Example, p. 58. In working numerical examples we may use the values 980 cm. per sec. per sec. or 32 feet per sec. per sec.

When the body is projected downwards and starts with velocity u the acceleration g is in the direction of motion, and

$$v = u + gt \quad\text{.......................(i)},$$

$$s = ut + \tfrac{1}{2}gt^2 \quad\text{.......................(ii)},$$

$$v^2 = u^2 + 2gs \quad\text{.......................(iii)}.$$

If the body be "dropped" it starts from rest so that $u = 0$.

If the body is projected vertically upwards and starts with the velocity u the acceleration g is opposite to the direction of motion and we have

$$v = u - gt \quad\text{.......................(i)}\ a,$$

$$s = ut - \tfrac{1}{2}gt^2 \quad\text{.......................(ii)}\ a,$$

$$v^2 = u^2 - 2gs \quad\text{.......................(iii)}\ a.$$

47. Problems on falling bodies.

(1) *To find the space passed over by a falling body in the* n*th second of its motion.*

Let s_1 be the space up to the beginning, s_2 up to the end of the nth second, then $s_2 - s_1$ is the space required.

Now, assuming the body to have been dropped,

$$s_1 = \tfrac{1}{2}g(n-1)^2,$$

$$s_2 = \tfrac{1}{2}gn^2.$$

$$\therefore\ s_2 - s_1 = \tfrac{1}{2}g\{n^2 - (n-1)^2\} = \tfrac{1}{2}g(2n-1).$$

(2) *A particle is projected upwards with velocity* **u**.

a. Find the height to which it will rise. Let this be H, the particle moves up till it reaches the height H, then it is instantaneously at rest and finally falls. Hence at a height H the velocity is zero,

$$\therefore\ 0 = u^2 - 2gH,$$

$$\therefore\ H = \frac{u^2}{2g}.$$

β. Find the time of rising. Let this be T_1, then at time T_1 the velocity is zero,

$$\therefore\ 0 = u - gT_1;$$

$$\therefore\ T_1 = \frac{u}{g}.$$

γ. *Find the time of falling.* Let this be T_2, then T_2 is the time of falling a height H.

$$\therefore\ \tfrac{1}{2}gT_2^2 = H = \frac{u^2}{2g},$$

$$\therefore\ T_2 = \frac{u}{g} = T_1.$$

Hence the times of rising and falling are the same. See also Example 4, p. 67.

δ. *Find the time at which it is at a height* h. Let this time be T_3.

Then $$h = uT_3 - \tfrac{1}{2}gT_3^2,$$

$$\therefore\ gT_3^2 - 2uT_3 + 2h = 0.$$

$$T_3 = \frac{u \pm \sqrt{u^2 - 2gh}}{g}.$$

Hence T_3 has two values, one corresponding to the upward passage, the other to the downward passage of the particle.

ϵ. *Find the velocity at a height* h. Let this be v.

Then $$v^2 = u^2 - 2gh.$$

Hence if $h = 0$, or the particle is on the ground

$$v^2 = u^2, \quad v = \pm u.$$

Thus the velocity with which the particle reaches the ground is equal and opposite to that with which it starts.

48. Composition and Resolution of Accelerations.

A particle may have two or more accelerations communicated to it simultaneously. We proceed to determine the resultant effect.

PROPOSITION 19. *To find the resultant of two accelerations.*

Acceleration is measured by the change in velocity per second. If then OA, OB, Fig. 49, represent two accelerations communicated to a body, OA and OB represent the changes which take place in the velocity of the body per second.

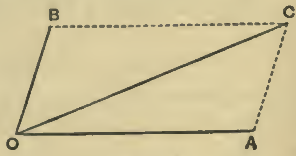


Fig. 49.

Complete the parallelogram $AOBC$. Then two velocities OA, OB have for their resultant OC. Thus the resultant change per second in the velocity of the body is OC. Hence OC is the resultant acceleration.

But OC is the diagonal of a parallelogram whose sides are OA and OB, the component accelerations. *Thus accelerations are combined according to the parallelogram law.* Hence the Propositions in Sections 30—31 about the Composition and Resolution of displacements apply to accelerations.

We may give *an alternative proof* of the above proposition thus.

Let PO, Fig. 50, represent the velocity of the body at any moment; let OA and OB represent the accelerations or changes per second which are to take place independently in the velocity. Draw AC equal and parallel to OB, then to find the velocity at the end of 1 second we have to combine with PO two velocities represented by OA and OB.

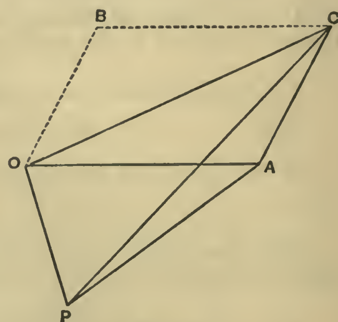


Fig. 50.

Combining PO and OA we get PA, combining with this OB we get PC. Thus PC represents the resultant velocity at the end of the second and PC is obtained by combining PO and OC. Thus OC is the change in velocity per second, that is, it is the resultant acceleration.

EXAMPLES.

MOTION WITH UNIFORM ACCELERATION.

1. Find the space traversed by a falling body in the eighth and tenth seconds of its motion.

2. With what velocity must a body be projected downwards in order to describe in 1 second a space equal to that described by a body falling freely in 2 seconds?

3. A bullet is shot up with a speed of 1000 feet per second, how high will it rise and after what time will it strike the earth again?

4. A body starts with a velocity 5 and has an acceleration of 2·5 in the direction of motion.

Find (i) its velocity after 3 seconds;
(ii) the space it has moved over in that time.

5. At the end of 3 seconds the acceleration of the body in the preceding question changes to 5 in a direction opposite to that of the motion; how far will it go before coming to rest?

6. A velocity of 5 is changed into one of 5 in a direction at right angles to itself, find the change in velocity and the acceleration supposing it uniform and that the change occurs in 10 seconds.

What will the velocity be at the end of 1 minute?

7. A stone is dropped over a cliff into water and the sound of the splash is heard after an interval of about $9\frac{1}{2}$ seconds; assuming the velocity of sound to be about 1150 feet per second, find the height of the cliff.

8. The velocity of a train passing two stations 1 mile apart is observed to be 30 and 50 miles an hour respectively; calculate its acceleration assuming it to be uniform.

9. A bullet shot up passes a point 1600 feet above its starting point on its upward and downward path respectively at an interval of half a minute, find its initial velocity and the height to which it rose.

10. A particle is moving under uniform acceleration with a velocity of 100 feet per second, at the end of 1 minute its velocity is 220 feet per second. How far will it move in 10 minutes and what will then be its velocity?

11. Draw the velocity-time curve for the bullet in question 9 and find hence or otherwise when it strikes the ground.

12. A body has a velocity of $3g$ and an acceleration g (a) in the direction of motion, (b) in the opposite direction. Find how far it moves in the first case before its velocity is doubled and in the second before it is halved. Find also the distances moved through in the two cases.

13. A body has an initial velocity 25 and an acceleration 1 opposite to the direction of motion. At what time will it have moved over half the distance it moves through before coming to rest and what will be its velocity then?

14. A body has an initial velocity u and an acceleration a. At what time after starting will it be moving with twice its initial velocity?

15. A stone is let drop from a given height and another is at the same instant projected vertically up to meet it. They pass at half the height, how high will the second stone rise and with what velocity does it start?

16. How long will a body, falling from rest, take to acquire a velocity of 96 feet per second?

17. With what velocity will a particle reach the ground if allowed to fall over a cliff 1156 feet in height?

18. A particle has an acceleration of 32 feet per second and passes a point at a distance of 1156 feet from the start with a velocity of 272 feet per second; find its initial velocity.

19. How high will a stone rise if projected up with a velocity of 250 feet per second?

20. A particle falls from a height of 78·48 metres, when will it reach the ground?

21. The speed of a train moving with uniform acceleration is doubled in a distance of 3 kilometres. It traverses the next $1\frac{1}{16}$ kilometres in 1 minute, find its initial speed and its acceleration.

22. A particle is thrown up with a velocity of 300 metres per second, how high will it rise and when will it strike the ground?

23. With what velocity will a particle reach the ground if it fall from a height of 400 metres?

24. The acceleration of a falling body is 981 cm. per sec. per sec.; find this in yards per min. per min.

25. How long will a falling body take to acquire a velocity of 100 metres per second?

26. The stick of a rocket reaches the ground 3 seconds after the explosion is seen, assuming the rocket to have been at rest when it burst, how high was it?

27. A train starts from rest and after 1 minute its speed is thirty miles per hour. Find the acceleration each second in feet per second supposing it uniform.

28. A moving point passes over 10 ft., 12 ft. and 16 ft. in three successive seconds, find its average velocity during the three seconds and determine whether or not it is moving with uniform acceleration.

29. The velocity of a body moving in a straight line is 32 feet per second at the end of 2 minutes and 40 feet per second at the end of 3 minutes. Find its initial velocity and its acceleration.

30. A train passes a station with a velocity of 30 miles per hour, and on passing the next, distant one mile, its velocity is 25 miles per hour. What is its acceleration?

31. A body moves in a straight line with uniform acceleration of 3·2 feet per second per second; find the time necessary to increase its velocity by a velocity of 15 miles per hour.

32. The position of a body moving with variable velocity is observed for each instant of its motion. Shew how to find from these observations the velocity and the acceleration of the body.

33. Prove that, when a body falls with a uniform acceleration, the difference between the square of the velocity at the beginning and end of the fall equals twice the product of the acceleration and the space traversed.

34. Two heavy bodies are dropped at the same time, one from a height of 50 feet, and the other from a height of 25 feet; find the height and velocity of the first, when the second touches the ground.

35. Prove that if a body is projected vertically upwards with the velocity of 64 feet per second, and 3 seconds afterwards a second body is let fall from the point of projection, the first body will overtake the second body one second and a half later at 36 feet below the point of projection, taking the acceleration of gravity to be 32 and neglecting the resistance of the air.

36. A balloon has been ascending vertically at a uniform rate for $4\frac{1}{2}$ seconds, and a stone let fall from it reaches the ground in $6\frac{1}{2}$ seconds after leaving the balloon: find the velocity of the balloon and the height from which the stone is let fall.

37. A particle moving with uniform acceleration in the direction of its motion has a velocity of 200 feet per second at the end of the third second, and of 260 feet per second at the end of the fourth second; what will be its velocity at the end of the fifth second? and what was the velocity at the instant from which the time has been reckoned?

38. A man on the bank of a river starts running with uniform speed just as the bow of a boat 60 feet long moving with uniform velocity is opposite to him. When the stern of this boat is opposite to him, the bow of a second boat, also moving with uniform velocity and of the same length as the first boat is just level with the first boat's stern, and just as the second boat passes the man, its bow is 20 feet behind that of the first boat. Find the distance apart of the boats when the man started.

39. Two trains of equal length, running at the rate of 40 miles an hour and 50 miles an hour respectively are approaching a level crossing. If the respective distances of the heads of the trains at the same instant from the crossing are 600 yards and 800 yards; find the greatest length that the trains can have so that there may not be a collision between them.

40. A train is moving at a rate of 60 miles an hour, and a gun is to be fired from a carriage window to hit an object which at the moment of firing is exactly opposite the window. If the velocity of the bullet be 440 feet per second, find the direction in which the gun must be pointed.

41. A man walks 2 miles in $\frac{1}{2}$ an hour, then drives 5 miles in $\frac{3}{4}$ hour, afterwards he travels 10 miles in an hour, and completes a further 8 miles of his journey in 20 minutes. Express his second speed in miles per hour, and his last speed in feet per second; and find what his average speed on the whole journey has been.

42. A man is running with a velocity of 6 miles per hour in a shower of rain which is descending vertically with a velocity of 11 feet per second. Find the tangent of the angle which the apparent direction of the rain makes with the horizon.

43. A certain mark on the circumference of a flywheel 6 ft. in diameter passes a fixed point three times every minute. Find the velocity of the mark and the angular velocity of the wheel.

44. A point moving in a straight line describes a space x in a time t, and its velocities at the beginning and end of the time are u and v. Find an expression for the mean acceleration of the point, and if its acceleration is constant, prove that $2x=(u+v)\,t$.

45. ABC is a triangle right-angled at C. Points start from A and B at the same instant and move towards C with uniform acceleration inversely proportional to AC and BC respectively. Shew that their least distance apart will be $\dfrac{AC^2 \sim BC^2}{AB}$,

46. A stone is thrown vertically upwards and returns to the point of projection after 5 seconds; find the greatest height to which it rises and its velocity, on its return, at the point of projection.

47. A stone is thrown vertically upwards and just reaches a height 80 feet above the point of projection. Find its velocity, on its return, at the point of projection and the whole time taken.

48. A uniformly accelerated body passes two points 6 feet apart in $\frac{1}{4}$ second; 4 seconds after reaching the first of these points the body had a velocity of 110 feet per second; find the velocity and acceleration of the body.

49. A falling body passes two points 10 feet apart in $\frac{1}{2}$ second: it subsequently passes two other points also 10 feet apart in $\frac{1}{10}$ second. If the acceleration due to gravity is 32 feet per second per second, find the distance between the first and the last of these four points.

50. If the measure of an acceleration is 528 when a yard and 6 seconds are the units of length and time, find its measure when a mile and an hour are the units of length and time.

CHAPTER IV.

MOMENTUM.

49. Mass. In the two following chapters we shall endeavour to obtain from experiment some notion of the meaning of the terms Mass and Force as used in Mechanics and to arrive at certain laws which express the results of the experiments. These laws, known as Newton's Laws of Motion, are from this point of view, generalizations from simple observations. Having arrived at these generalizations, we can start afresh and, assuming the laws as true in all cases, can deduce from them the motion of bodies under various complex circumstances. This is done in Chapter VI. and the following chapters, in which Mechanics is treated as a Deductive Science based on Newton's Laws as Axioms.

We are now about to consider certain effects produced by a moving body which we may treat as a particle. These effects we shall find may be different for different bodies; from the consideration of them we may obtain a definite idea of the meaning of the term **Mass** in Mechanics.

We can recognize in bodies in many ways a property which depends partly on their size and partly on the substance of which they are composed. Thus, if we take two balls of iron of considerably different sizes and hang them up by long strings of the same length, a very slight effort is sufficient to give a considerable velocity to the small ball, while a strong push is needed to displace the large ball appreciably; the two balls are said to differ in mass. Or again consider two casks of the same size, the one filled with sand, the other with feathers; a slight kick is sufficient to start the second cask rolling, a vigorous shove will hardly stir the former; we say that the mass of the sand is greater than that of the feathers.

A heavy flywheel properly mounted on ball bearings continues to rotate for a long time when set in motion, an

appreciable effort is needed however either to stop it, or to start it when at rest; the flywheel is said to have mass, and the greater the mass the greater the effort needed to stop it in a given time. Two identical lumps of metal when suspended by a fine string over a light pulley remain at rest; a downward push applied to one will start them moving, and when started they continue to move, but a stronger effort is needed if the bodies suspended be large than is required if they be small.

These and similar observations lead us to recognize that property of bodies to which the name of **Mass** has been given. We are able as we shall see to compare the masses of two bodies and shall find that for a given homogeneous substance the mass of a body depends on its volume, while for bodies of given volume the mass depends on the substance of which the bodies consist and on its physical state.

50. Experiments on the Measurement of Mass.

Newton describes in his *Principia* certain experiments on the collision or impact of bodies from which he draws several important conclusions. The observations just described lead us to the conception of mass as a fundamental notion; experiments based on those of Newton enable us to give definiteness to the idea.

Newton in his experiments employed two spherical balls suspended from two points in the same horizontal line by parallel strings of such lengths that when at rest the balls were in contact and their centres were at the same distance below their points of suspension (Fig. 51). The experiments consisted in drawing the balls apart to various small distances, and then allowing them to fall simultaneously; the balls then struck each other at the lowest point of their swing, and the velocities with which they impinged were calculated[1]; the positions to which the balls rose after impact were observed and from this observation their velocities after impact were obtained; from the relation between these velocities various important deductions can be drawn.

Fig. 51, taken from the *Principia*, shews the arrangement adopted by Newton. A body such as the ball A or B, suspended so as to be able to swing backwards and forwards

[1] See Section 146.

about its position of rest is called a simple pendulum, and in the deductions from the experiments we make use of two laws discovered by Galileo, as to the motion of a simple pendulum[1].

According to the first law, if the ball of a simple pendulum be pulled aside a moderate distance, so that the string is

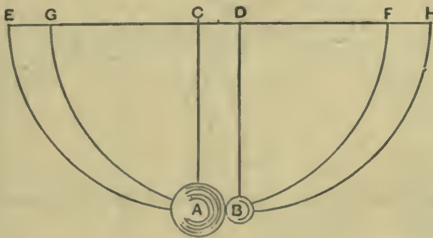


Fig. 51.

inclined to the vertical at a small angle, when the ball is released it will take very approximately the same time to reach its equilibrium position, in which the string is again vertical, whatever be its starting point, provided only that the angle the string makes originally with the vertical be not large. Thus if the two balls A and B be drawn a short distance apart and let fall simultaneously, since the distances between the points of suspension and the centre of each ball are the same, they will always impinge at the lowest point.

If the ball A be drawn a very short distance aside and released, its velocity as it passes through the equilibrium position will be small; if the original displacement be larger, the ball after release will arrive at its lowest point in the same interval of time as before, but since in the second case it has in that interval traversed a greater distance than in the first case the velocity with which it reaches it will be greater.

The second law referred to above enables us to calculate what the velocity is.

Let P, Fig. 52, be the point from which the ball A is allowed to start. Join PA. Then it can be shewn that the velocity with which the ball will reach its equilibrium position is

[1] Experiments to verify the laws are described later (see § 130).

proportional to the distance PA. If the ball A, after striking the second ball, rise to Q its velocity is proportional to AQ, we

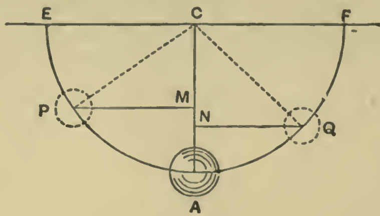


Fig. 52.

can compare the velocities before and after impact by measuring the lines PA and QA.

Let PM and QN be horizontal lines drawn from P and Q respectively to meet the vertical line CA in M and N. Then if the arcs PA and QA be small, the ratio of PA to QA is nearly the same as that of PM to QN. The velocities before and after impact are approximately proportional to PM and QN.

We could perform our experiments with the apparatus used by Newton, an arrangement however which has been devised by Professor Hicks of Sheffield will serve better. It is called by him a Ballistic Balance.

51. Hicks' Ballistic Balance. The apparatus is shewn in Fig. 53. It consists of a rectangular wooden framework $ABCD$, about 100 cm. long and 125 cm. high. The bar AB is horizontal, AC and BD are vertical, and the framework is arranged so as to stand securely on a table. Four parallel bars EF about 20 cm. long can be adjusted across AB.

From these bars two carriers G, H, are supported by fine wires or threads as shewn in the figure, the carriers are small rectangular pieces of wood, the lengths of the wires and the positions of the bars EF are adjusted so that the planes of the carriers as they swing are always horizontal, and the two carriers strike each other perpendicularly when each is at the lowest point of its swing. The ends of the carriers which come into contact are fitted with some sharp-pointed pins so that the carriers after impact adhere together.

LMN is a horizontal bar fitted below the carriers and carrying two scales, one for each carrier. A vertical pointer is fixed to each carrier and moves over the corresponding scale, which is adjusted so that the pointer reads zero when the carriers are in contact. If a carrier is pulled aside and let

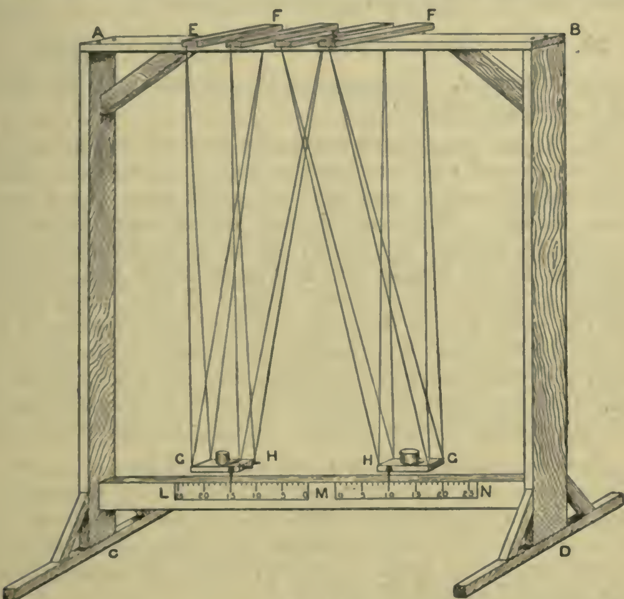


Fig. 53.

go, the velocity with which it reaches its lowest point is proportional to the horizontal distance through which it has been displaced, and this is given directly by the scale reading of the pointer at the starting point. The actual value of the velocity will depend on the dimensions of the instrument; if the vertical distance between the carrier and the points of support of the wires be 109 centimetres, a displacement of 1 centimetre along the scale can be shewn to give rise to a velocity of 3 cm. per second; each centimetre of the scale corresponds therefore to a velocity of 3 centimetres per second.

. By attaching a string to each carrier, it can be pulled aside any required distance, on releasing the strings simultaneously the two carriers are set in motion and impinge at the lowest point of their swings. By passing the strings over pulleys they can both be fastened to the same clip so as to secure that the two carriers start simultaneously.

52. Experiments with the ballistic balance.

EXPERIMENT 9. *To determine the condition that two bodies may have equal masses.*

(*a*) It has been stated previously that the masses of two equal volumes of any homogeneous material are equal. Take two equal volumes of lead, say two cubes some 3 or 4 centimetres in edge. Place one on each carrier and displace the two carriers equally, say 2 cm.; release them simultaneously; they meet at the bottom and it will be found that both are reduced to rest. If two equal volumes of iron, silver, etc. be used the result will be the same. Repeat the experiment but let the displacement be greater, still keeping it the same for both; the result is always the same, the masses are reduced to rest. Thus two identical lumps of matter when they impinge on each other directly with the same velocity are reduced to rest.

Now take a lump of iron and a lump of lead of the same volume, place one on each carrier and repeat the experiment; the impact will no longer result in rest, the lead lump will continue to move forwards though with reduced velocity, the iron will be driven back. The iron is said to have less mass than an equal volume of lead.

(*b*) Take a smaller piece of lead and repeat the experiment, the velocity after impact will be less than before—with a still smaller piece of lead the velocity may even be reversed. By adjusting the volume of the lead we can again obtain the result that there is no motion after impact. When this has been secured the two masses of iron and lead are equal.

DEFINITION OF EQUAL MASSES. *The* Masses *of two bodies are* equal, *if, when the bodies impinge on each other directly with equal velocities, they are reduced to rest.*

53. The comparison of Masses. If the volume of the lead used in EXPERIMENT 9 (*b*) be measured it will be found to be about 78/114 of that of the iron.

Thus, according to this definition of equal masses, a given volume of iron has the same mass as 78/114 of that volume of lead. Hence the masses of equal volumes of iron and lead are as 78 to 114. We have thus a means of comparing the masses of two bodies.

In this experiment the bodies impinge with equal velocity, they are reduced to rest, and we say that their masses are equal.

Consider now what happens if the velocities with which the bodies meet be not equal.

Place on the carriers two equal lumps of lead. Draw back one carrier further than the other and release them. On impact the carrier which has the greater velocity continues to move onwards, the motion of the other is reversed. Replace one of the lumps of lead by an iron lump of equal volume and draw it back further than the lead lump so that on impact the iron lump may have the greater velocity; the velocity after impact will be less than it was when the two met with the same velocity, and by careful adjustment positions can be found for the carriers such that they remain at rest after impact.

Thus for example if the lead lump be displaced 7·8 cm.[1], and the iron 11·4 cm., then after impact the carriers will be at rest, the same will be true if the displacements respectively be 3·9 and 5·7 cm. or 15·6 and 22·8 cm.

Now these numbers are to each other respectively in the ratio of 78 to 114, that is, in the ratio of the mass of the iron to the mass of the lead.

Again, the velocities with which the two impinge are proportional to the displacements, and we find in this case that there will be rest after impact provided that we satisfy the relation given by the equation,

$$\frac{\text{Final Velocity of Lead lump}}{\text{Final Velocity of Iron lump}} = \frac{\text{Mass of Iron}}{\text{Mass of Lead}},$$

[1] In these numbers no allowance has been made for the mass of the carriers. Should this be appreciable compared with the masses of the bodies placed on them some correction will be required.

or writing u_1, u_2 for the two velocities, m_1, m_2 for the two masses, provided that

$$\frac{u_1}{u_2} = \frac{m_2}{m_1},$$

or
$$m_1 u_1 = m_2 u_2.$$

Whenever then this relation is satisfied the carriers after impact will come to rest.

Thus we can use the ballistic balance to compare two masses by determining the velocities with which they must impinge directly in order to be reduced to rest, for we have the result that, *When two bodies are caused to impinge directly so as to adhere together and are reduced to rest by the impact their masses are inversely proportional to the velocities with which they impinge.*

Moreover we can shew, by direct experiment, that the results are not modified by altering the shape of the lumps of matter used. If we determine the ratio of the masses of a lump of iron and a lump of lead we may alter the shape of the lead by hammering it, or in any other way. If we do not remove any of the lead its mass as determined by the ballistic balance in terms of that of the iron lump remains unchanged.

Other and simpler methods of comparing masses will be given later. (*Statics*, § 59.)

54. The Unit of Mass. We have explained § 11 that we assume as the unit of mass the mass of a certain lump of platinum called a kilogramme. We may imagine then that we use this in one of the carriers of the ballistic balance, and give it a certain velocity; we can determine in terms of this the mass of any other body by finding the velocity which must be given to that body in order that it may be reduced to rest by impact with the standard mass.

Moreover the ratio of the masses of two bodies is found to remain the same from day to day; and is not altered by carrying the bodies from point to point of the earth's surface. The mass of any given body has at all times and places a constant ratio to that of the standard; if we assume its mass to be a definite property of the standard then the mass of every other body is a definite property of that body.

55. Mass and Quantity of Matter. Suppose we determine by the ballistic balance the mass of a body; remove part of the body and again measure its mass; it will be found that the mass is reduced.

Now in ordinary language we should say that we had taken away some of the matter of which the body is composed. We thus see how it comes about that mass is looked upon as measuring the quantity of matter in a body, and what is meant by the statement, that the mass of a body is the quantity of matter of which it is composed.

It may however be convenient sometimes to employ the term "quantity of matter" as identical with the term "Mass," to say that in Dynamics the quantity of matter in a body is measured by its mass; the mass can be compared with the unit mass by means of a ballistic balance or in some equivalent manner.

56. Momentum. The experiments with the ballistic balance have led us to recognize a quantity in mechanics which depends on the product of the mass of a moving body and its velocity. The motion which a given body can communicate by impact to another body depends on this quantity. This quantity is called **Momentum** and we shall find that it is of fundamental importance.

DEFINITION. *The Momentum of a body is the product of its mass and its velocity.*

Thus if a mass of m grammes be moving with a velocity of v centimetres per second its momentum is mv.

The unit of momentum therefore is the momentum of a mass of 1 gramme which moves with a velocity of 1 cm. per second.

Various names have been suggested for the unit of momentum but none of them has received general acceptance. When then, we say that the momentum of a body is 10 we mean that it has 10 units of momentum; its momentum therefore is the same as that of a mass of 10 grammes moving with a velocity of 1 cm. per second or of 1 gramme moving with a velocity of 10 cm. per second, or of x grammes moving with a velocity $10/x$ cm. per second.

In dealing with momentum we must remember to take into account the direction in which the body is moving. Thus a billiard ball which impinges directly on the cushion has the direction of its motion reversed by the impact; if we agree to call its velocity and its momentum positive before the impact, we must call them negative afterwards.

57. Condition of rest after Impact. We can now express in terms of momentum the condition that two bodies which meet directly and adhere should be brought to rest by the impact. From Section 53 we know that the condition is that

$$m_1u_1 = m_2u_2.$$

Now m_1u_1 is the momentum of the first body, m_2u_2 that of the second; the condition then for rest is that the momenta of the two bodies should be equal in magnitude and opposite in direction. If we call the momentum of one body positive that of the other will be negative, we may say then that the condition for rest after impact is that the total momentum of the system before impact should be zero.

The momentum after impact is zero, so that we see that in this case there is no change in the momentum of the system.

***58. Further experiments with the ballistic balance.** By the previous experiments we have determined the condition that the bodies should be reduced to rest by the impact. We now wish to determine the velocity with which they will move if this condition be not fulfilled, we still suppose that the two carriers adhere together after the impact.

EXPERIMENT 10. *Two masses impinge directly and adhere; to determine by experiment the relation between their velocities before and after impact.*

Take two equal masses, such as the two lumps of lead used in EXPERIMENT 9 and place one in each carrier of the balance. Let one mass remain at rest in its lowest position. Displace the second carrier a measured distance a, say 20 centimetres, and let it go. After the impact the two carriers move to-

gether. Observe the extreme distance to which they swing; let it be b centimetres. Then it will always be found that

$$b = \tfrac{1}{2}a,$$

thus if the original displacement of the first mass be 20 cm. the joint displacement after impact will be 10 cm.

Now the first displacement measures the velocity with which the one mass strikes the other, the second measures the velocity with which the two masses move after impact. We thus see that in this case the velocity is halved by the impact, but at the same time the moving mass is doubled, the momentum therefore of the system remains the same, the one mass loses momentum while the other gains an equal amount, there is no change in the total amount, it is distributed between the two instead of being entirely in the one.

Now, however the masses on the carriers be changed, it will be found that this law always holds; the total momentum is unaltered in all cases.

If m_1, m_2 be the masses, u_1 the velocity with which m_1 strikes m_2 at rest, and u the common velocity after impact, then we shall find that

$$(m_1 + m_2)\,u = m_1 u_1,$$

the momentum of the two after impact is equal to the sum of the momenta before; the velocities u_1 and u are measured respectively by the original displacement of the first mass and the joint displacement b of the two after impact. Make a series of observations of b giving a different values such as 2, 4, 6, 8, 10, etc. cm. It will be found in all cases that the ratio of b to a is a constant and that this ratio is equal to the ratio of m_1 to the sum $m_1 + m_2$.

Make another series of observations in which both the masses m_1 and m_2 are in motion before impact. Displace m_1 a distance a_1 and m_2 a less distance a_2 in the same direction; on releasing the two simultaneously m_1 will acquire a greater velocity than m_2 and will overtake it at the bottom of the swing; after impact the two bodies will move on together with the same velocity u and it will be found that the momentum of the two is the sum of the original momenta so that

$$(m_1 + m_2)\,u = m_1 u_1 + m_2 u_2,$$

or, if b be the first displacement after impact

$$(m_1 + m_2)\, b = m_1 a_1 + m_2 a_2,$$

$$b = \frac{m_1 a_1 + m_2 a_2}{m_1 + m_2}.$$

Make a series of observations for different values of a_1 and a_2 and verify this formula.

A similar result can be obtained when the two bodies move in opposite directions before they impinge, only in this case we must treat the momenta of the two as opposite in sign. The combined system will after impact move in the direction of motion of that body which before impact had the greater momentum; we shall have the equation

$$(m_1 + m_2)\, u = m_1 u_1 - m_2 u_2,$$

satisfied

or

$$b = \frac{m_1 a_1 - m_2 a_2}{m_1 + m_2}.$$

In all the above cases we see that there is no change of momentum produced on the whole.

***59. Impact of elastic bodies.** This same result is true and can be verified by the ballistic balance even though the carriers rebound from each other after impact; the observations[1] are rather more troublesome because there are two quantities b_1 and b_2 the displacements of the two carriers to observe.

We shall find that it is impossible without further knowledge as to the properties of the material to calculate *a priori* the values of b_1 and b_2 from a knowledge of the original displacements and the masses, but it will follow the results of observation that the values of b_1 and b_2 are always such that the momentum of the system remains unchanged.

They satisfy the relation

$$m_1 b_1 + m_2 b_2 = m_1 a_1 + m_2 a_2,$$

[1] In such experiments the pins or clips arranged to fasten the carriers together after impact are removed; the carriers therefore are free to rebound and the nature of the rebound will depend on the material of which they are composed as well as on their original momenta.

so that, if we call v_1, v_2 the velocities after impact which correspond therefore to b_1 and b_2 and u_1, u_2 the velocities before impact corresponding to a_1 and a_2, we have

$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2.$$

Thus when two bodies impinge directly the momentum remains unchanged.

Newton's experiments already referred to were designed in part to verify this law, in part also to determine another relation between v_1 and v_2 from which, when combined with the above, it might be possible to calculate v_1 and v_2. We shall return to this point again, for the present we are concerned with the law of the permanence or conservation of momentum.

60. Change of Momentum—Impulse. In each of the above experiments, while the momentum of the whole system has remained unchanged the momentum of each body has been altered, there has been a transference of momentum from one body to the other, the one has gained what the other has lost.

Thus, if in EXPERIMENT 10 we take two equal masses m, the velocity of the striking mass is changed by impact from u to $\frac{1}{2}u$; its momentum was mu and it becomes $\frac{1}{2}mu$; its loss of momentum therefore is $\frac{1}{2}mu$. The velocity of the second mass is changed from 0 to $\frac{1}{2}u$; its momentum therefore alters from zero to $\frac{1}{2}mu$, its gain is $\frac{1}{2}mu$ which is equal to the loss of the first mass.

Now this law is quite general, for with the notation of Section 58, the loss of momentum of m_1 is

$$m_1u_1 - m_1v_1,$$

or

$$m_1(u_1 - v_1).$$

The gain of momentum of m_2 is

$$m_2v_2 - m_2u_2,$$

or

$$m_2(v_2 - u_2).$$

Now we have

$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2.$$

Hence

$$m_2v_2 - m_2u_2 = m_1u_1 - m_1v_1,$$

or

gain of momentum of m_2 = loss of momentum of m_1.

Thus momentum is transferred unchanged in amount from the one mass to the other.

The name **Impulse** has been given to the whole change of momentum of a body.

DEFINITION. *The gain of momentum of a body is called* **Impulse.**

Thus in the experiments in Section 56 the second mass *gains* an amount of momentum $m_2(v_2 - u_2)$, its Impulse I_2 therefore is given by

$$I_2 = m_2(v_2 - u_2).$$

The first mass *loses* an amount of momentum $m_1(u_1 - v_1)$, its impulse I_1 is given by

$$\begin{aligned} I_1 &= -m_1(u_1 - v_1) \\ &= m_1(v_1 - u_1), \end{aligned}$$

and in this case I_1 is negative.

Moreover we have the result that

$$I_2 = -I_1$$

or

$$I_1 + I_2 = 0,$$

that is, the total impulse is nothing.

61. Transference of Momentum. Thus we have learnt from these experiments to consider momentum as a property of a moving body which we can measure in a definite manner. This property can be communicated through impact by one body to another; when such transference takes place under circumstances similar to those of the above experiments, there is neither loss nor gain of momentum, the one body gains what the other loses, the amount of momentum transferred is unaltered. Now we find that this law is of general application; momentum can be transferred from one body to another

in other ways than by direct impact; whenever such action goes on in an unimpeded manner there is no loss in the amount of momentum transferred, the gain of the one body is equal to the loss of the other.

The words "in an unimpeded manner" above are of importance. Consider two isolated particles and suppose that there are no other bodies near which can in any way affect their mutual action, this action is unimpeded; the two particles will move towards each other in such a way that in any given time they each gain equal amounts of momentum in opposite directions. The total change of momentum will be zero, the impulse of the one particle is equal and opposite that of the other.

But we cannot in practice secure that the above conditions shall be satisfied; it is impossible to obtain two particles free from the influence of all other bodies. Our experiments must be performed in the presence of the earth, and this may have an effect on the change of momentum produced. In the cases of impact with which we have been dealing no such effect is produced as we shall see later, the action is unimpeded.

Take another example. If a stone be held at a distance from the earth it has no momentum relatively to the earth; on releasing it it falls and acquires momentum, there is here a gain as far as we can observe of momentum; in reality we believe that there is no gain, for the earth has acquired momentum in the opposite direction equal to that of the stone.

We cannot of course verify this by direct experiment, we have no means of determining whether the earth moves towards the stone or not; the velocity acquired by the earth would be excessively small, for its mass is enormous compared with that of the stone.

Thus if a stone fell from a height of 5 metres it would on reaching the ground have a velocity of about 1000 centimetres per second ($v^2=2gs=2\times 981\times 500$, $v=1000$ approximately). Suppose the mass of the stone to be 1 kilogramme or 10^3 grammes; the mass of the earth is about 5×10^{27} grammes. Thus the momentum of the stone is $10^3\times 10^3$ or 10^6 units of momentum. Hence the velocity of the earth is $10^6/5\times 10^{27}$ or 2×10^{-22} cm. per second. Now there are rather more than 3×10^7 seconds in a year; hence if the earth were to continue to move for a year with the velocity thus acquired it would only traverse 6×10^{-15} or ·000,000,000,000,006 of a centimetre.

62. Conservation of Momentum. We may sum up then the results of the experiments and our discussion of them with the statement that by the mutual action between two bodies momentum can be transferred from the one to the other. When this mutual action is unimpeded the momentum transferred remains unchanged in amount.

This principle is known as the Conservation of Momentum.

CHAPTER V.

RATE OF CHANGE OF MOMENTUM. FORCE.

63. Force. The experiments described in the last chapter have shewn us how to attach a definite meaning to the term mass as used in Mechanics and have led us to recognize Momentum as a fundamental property of a moving body. We have seen also that momentum is transferred without loss from one body to another by impact and have given a name, Impulse, to the change in momentum.

Now the velocity of a body is uniform and its momentum constant when it moves in a straight line and passes over equal spaces in equal times. If we observe the velocity of any body[1] by noting its positions at given intervals of time, we find that there are very few cases in nature in which the velocity is uniform; in nearly all the velocity is variable, the body has acceleration.

Thus in most cases the momentum of a moving body changes. We are now about to investigate in certain cases the rate at which this change takes place. This rate of change of momentum has received a name, it is called Force.

DEFINITION. *The Rate at which the Momentum of a moving body changes is called* Impressed Force.

In the experiments just described the change of momentum has been a sudden one. The Impulse has occurred in a very

[1] As has been already explained, § 15, we are at present dealing only with bodies which for the purposes of our investigation may be treated as particles.

brief interval of time; suppose now we consider a series of very small impulses applied for brief consecutive intervals of time; the small changes of momentum occurring during each impulse will add up; at the end of a finite time a finite change of momentum is produced, but the change has been a gradual one, the motion has taken place under an impressed force.

Force is often looked upon as something external to the body acting on it and causing it to move. Now when we say that a body is moving under the action of a force all that we can observe is a change in the momentum of the body. We are however often not content with the simple observation that the motion of the body is changing in a definite way, we endeavour to assign a cause for this change of motion and call this cause Force. We look upon the change in momentum as due to the mutual action between the moving body and some other body, the Earth for example, which can influence it, and we say that the change of momentum is due to this action. Our sensations give us some knowledge of a mutual action between ourselves and other bodies which if not impeded is followed by motion, and it was doubtless to this muscular sense that the idea of force was originally attached. But our sensations alone cannot enable us to measure Force. We cannot prove that Force as measured by the rate of change of momentum corresponds to our muscular sensations. For our purposes we therefore dismiss at once the notion of there being any connexion between the two. Force as a *Cause of Motion* we have not here to consider; it will suffice for us to define it as *Rate of change of Momentum* and proceed to examine certain simple cases of motion with a view of seeing what deductions we can make from them as to the relation between Force and Motion.

In this we are adopting a historical method of procedure. The experiments we are about to describe resemble those by which Galileo established some of the fundamental laws of Mechanics; their discussion will lead us naturally to the consideration of Newton's Laws of Motion given by him in the first pages of the Principia as the fundamental Axioms of the subject.

We are about to deal with the rate of change of momentum of bodies and to consider in the first case that of falling bodies; we shall shew how to express the rate of change of momentum of a body in terms of its mass and its acceleration.

64. Measurement of Force.

PROPOSITION 20. *The Rate of change of momentum of a body is the product of its mass and its acceleration.*

Let the velocity of the body initially be u cm. per second, and suppose that after t seconds it is v cm. per second, and that the acceleration is uniform and equal to a cm. per second per second.

Then the momentum originally is mu, after t secs. it is mv. Thus

The change of momentum in t seconds is $mv - mu$ and the change in 1 second is $(mv - mu)/t$ or $\frac{m(v-u)}{t}$.

But $$v = u + at.$$

Hence $$\frac{v-u}{t} = a,$$

and the change in momentum per second is ma.

But when a quantity changes uniformly the change per second measures the rate of change.

Thus if F be the impressed force we have $F = ma$.

Hence the rate of change of momentum is ma.

When the acceleration is variable the same expression holds, for we deal with variable acceleration by supposing it uniform for a very short space of time and considering what takes place in the limit when the time is indefinitely diminished; now the above formulæ are true when t is indefinitely small and a variable.

65. The acceleration of a falling body. When a body falls it moves with a continually increasing velocity: the motion is accelerated. Observation shews as we have already stated that the acceleration is a uniform one. This acceleration is spoken of as the acceleration due to gravity; its value in England is about 981 cm. or 32·2 feet per second per second. The most direct way of proving this would be by observing the distance a body falls through in various intervals of time, this method is not easy to put into practice, for unless the intervals of time are very short the space traversed and the velocities acquired become considerable and difficult of measurement, while if the times are short it is difficult to measure them exactly.

The difficulty was avoided by Galileo, who observed the time taken by a ball to roll down an inclined plane, and from that inferred what would happen if it fell freely; it is met in another way, as we shall see shortly, in Atwood's Machine. We will however first give some experiments on bodies falling freely.

In experiments on Motion we need some arrangement for measuring short intervals of time. For many purposes a good stop-watch will serve; this is fitted with a long seconds-hand, the dial is divided into seconds and these subdivided into fifths and observations can thus be made to the fifth of a second.

In other cases a pendulum which ticks once a second is useful. It is not necessary that there should be any clockwork attached. A heavy pendulum once started will continue to move sufficiently long for an experiment without the aid of a spring.

It is convenient for many purposes to have an arrangement attached to the pendulum, by which a circuit carrying an electric current may be made or broken—as is most convenient—once a second. The current may be made to ring a gong which thus sounds at intervals of a second

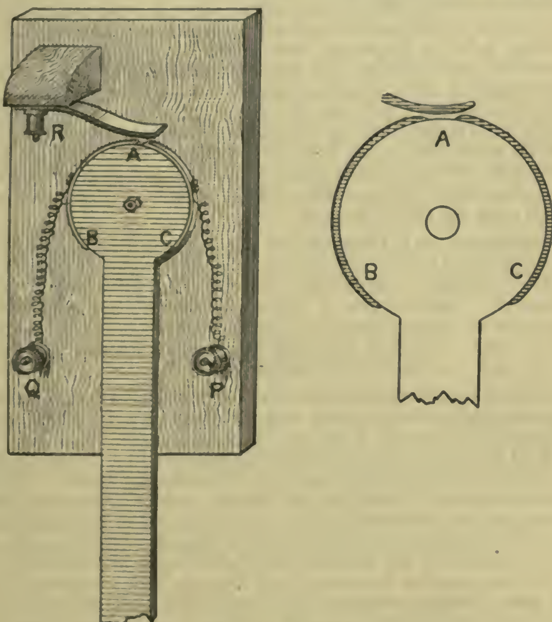


Fig. 54.

and marks the time more definitely than the ticks. Fig. 54 shews a device which is useful.

The end of the pendulum is an arc of a circle, on the edge of this, at A the top of the arc are two strips of thin brass insulated from each other. These are connected by flexible wires to two binding screws P and Q. A strip of thin brass connected to a binding screw R makes contact with the top of the pendulum rod at a point just above its point of support. A strip of paper or thin insulating material (shewn in black in the section at the side of Fig. 54) is pasted over the strip of brass AB leaving only a narrow portion of the brass near A exposed. The screw R is connected to the battery, P to the electro-magnet and Q to the bell. The second poles of the electro-magnet and of the bell are connected together and to the second pole of the battery.

When the pendulum is pulled to the right contact is made between R and P by means of the spring and the strip AC; the current passes through the electro-magnet which is magnetised and can support an iron ball or a wooden ball which has a small piece of wire attached. When the pendulum is released this contact is maintained until the bob reaches the lowest point of its swing, and at the moment contact is broken between the spring and AB it is made between the spring and AC; the bell is rung and as it rings the ball drops; the contact with AC, since the brass exposed is very narrow, is only maintained for a very brief time, the gong sounds once or perhaps twice and then is silent until the pendulum again passes through its lowest position; the time of swing of the pendulum can be adjusted by altering the position of the bob and we thus have a means of marking seconds, half-seconds or other intervals after the ball is dropped.

A tuning-fork or a vibratory bar may also be employed to measure small intervals of time. The prongs of the fork when it is struck vibrate and each vibration occupies the same time; the period of vibration can be measured. Suppose now a light metal style is attached to one prong and a piece of smoked glass is held so that the point of the style is just in contact with it, if the glass be raised or lowered vertically, the style will trace a straight line along it.

Let us now suppose the fork is set in motion so that the vibrations take place in a horizontal plane and the glass moved uniformly past it, the straight line becomes a regular undulatory curve. Each loop of the curve is of the same size and it cuts the straight line drawn by the fork when at rest in points which are at equal distances apart. If the fork makes 20 complete vibrations per second each of these distances will correspond to the one-fortieth of a second and will measure the distance traversed by the plate in the fortieth of a second. If the motion of the plate be not uniform the distances will not be equal and the loops of the curve will not be alike, each space will however be the distance traversed by the plate in the corresponding fortieth of a second. By measuring the spaces we can deduce various consequences as to the motion.

EXPERIMENT 11. *To shew that a body falling freely passes over approximately* 490·5 *centimetres*—16·1 *feet—in the first second of its motion from rest.*

The seconds pendulum just described is used for this, the bob is adjusted so that the pendulum makes one complete oscillation in two seconds, it thus passes through its equilibrium position once a second and at each transit the gong is sounded once. The binding screws P and R are connected with an electromagnet and battery and the pendulum is drawn aside and held by a string so that connexion is made between P and R through the brass strip AB and the spring.

The electro-magnet which is thus magnetized and supports an iron ball is attached to a light wooden frame which can be raised by means of a string to any desired height; a measuring tape is also attached to the frame and the height to which it is raised can be easily measured.

Start the pendulum; then, as the pendulum reaches its lowest position the electric circuit round the magnet is broken, thus releasing the ball, and the gong is sounded at the same time. Raise the electro-magnet and adjust its height until the ball on falling strikes the floor simultaneously with the second stroke of the gong; this coincidence can be estimated with considerable accuracy. Measure the height of the electro-magnet; the interval between the two sounds is one second and during this interval the ball has fallen through the height just measured.

It will be found on making the measurement that the height is about 490·5 centimetres or 16·1 feet.

If we attempt to use this method to find the distance fallen through in a longer time, say 2 seconds, we find that the distance is too great for measurement in the Laboratory—it would be $4 \times 490 \cdot 5$ or 1962 cm. In order then to find how the space traversed by a body falling freely varies with the time we must have recourse to some method of measuring small intervals of time such as that described above.

EXPERIMENT 12. *To shew that the space passed over by a body falling freely from rest is proportional to the square of the time of motion.*

A massive tuning-fork, Fig. 55, making say 20 vibrations per second is mounted so that its prongs vibrate in a horizontal

plane and a light style is attached to one prong. The style may conveniently be a bristle from a brush and its end should point downwards. A glass plate of considerable mass

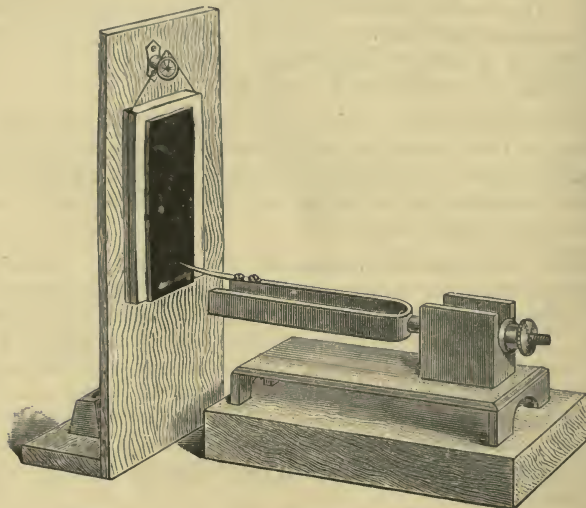


Fig. 55.

is supported, with its lower edge just below the point of the style, by a single string passing over a pulley and the plate can be allowed to drop by burning the string above the pulley. The plate is supported in such a way that the upper part of its front surface is slightly tilted forwards; hence as the plate falls it comes almost immediately into contact with the style. The back of the plate rests against two narrow vertical strips of wood; it is thus prevented from swinging in its fall. The front of the plate is coated with lamp black and the style as the fork vibrates marks a sinuous trace on the falling plate.

The style, if the fork had been at rest, would have traced a vertical line on the plate. This line can be drawn in

afterwards and a trace such as that shewn in Fig. 56 is obtained; the point A which was opposite to the style before the plate was started is also marked.

In such a case the spaces AB, AC, AD etc. represent the distances traversed in 1, 2, 3 etc. twentieths of a second and these distances can be measured; it will be found that approximately

$$AC = 4\ AB = 2^2 AB$$
$$AD = 9\ AB = 3^2 AB$$
$$AE = 16 AB = 4^2 AB,$$

and these are the spaces traversed in 2, 3, 4, etc. twentieths of a second. Thus the spaces traversed are proportional to the squares of the time.

We have already seen that when a particle moves with uniform acceleration it traverses spaces which are proportional to the squares of the times. Hence we infer that a falling body moves with uniform acceleration.

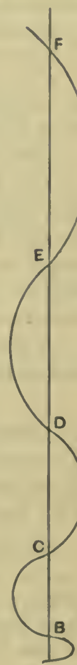


Fig. 56.

In making the experiment it will not usually happen that the trace passes accurately through the point A. This will affect the measurements of the distance traversed, but except in the case of the first few intervals the error will not be large. We can eliminate it from the result and use the experiment to shew that the acceleration is uniform by a method due to Prof. Worthington. For let l_1 and l_2 be the distances traversed in any two equal consecutive intervals of time t, then if the acceleration is uniform l_1/t and l_2/t will be the velocities at the middles of these two intervals. Thus the increase of velocity during the time t is $(l_2 - l_1)/t$ and the acceleration if uniform is $(l_2 - l_1)/t^2$. By making the calculation for various parts of the trace it is found that the same value is obtained from all for the acceleration, and this value is approximately 981 cm. per sec. per sec. It is thus proved that a falling body moves with uniform acceleration.

EXPERIMENT 13. *To shew by observations on falling bodies that in a given locality the acceleration* g *due to gravity is the same for all bodies.*

(*a*) Take two balls of different mass and drop them simultaneously from a height; they will reach the ground at

practically the same moment, they have moved over the same distance in the same time and kept together throughout their course, they have moved with the same velocity, their acceleration has been the same.

An experiment such as the above was first performed by Galileo about 1638. He dropped two shot of different masses from the leaning tower of Pisa and found that they reached the ground together.

The following is a more accurate form of the experiment.

(*b*) Fit two electromagnets on to a wooden frame and connect them so that the same current traverses the two. Pass a current round them and suspend a small iron ball from one, a larger ball from the other; then raise the frame and balls to the top of the room, on breaking the current the balls drop simultaneously and reach the floor together.

Thus two bodies of the same material but of different mass fall at the same rate.

(*c*) Repeat the experiment using for one of the iron balls a wooden ball into which an iron screw or nail has been fixed. By means of this the ball can be held up by the electromagnet. When dropped it will reach the floor very nearly simultaneously with the iron ball.

Hence two balls of different material fall at the same rate.

(*d*) If for the wooden ball a very light ball be substituted it will probably be a little longer in its fall than the iron ball. This is due to the resistance offered by the air to the passage of the balls. In a vacuum the time of fall of any two objects is the same. This can be proved by the aid of a piece of apparatus shewn in Fig. 57. A light and a heavy object—a feather and a sovereign—are placed on two little platforms at the top of a tall glass jar open below. The platforms can be released from outside the jar and the objects allowed to fall. The bottom of the jar is ground

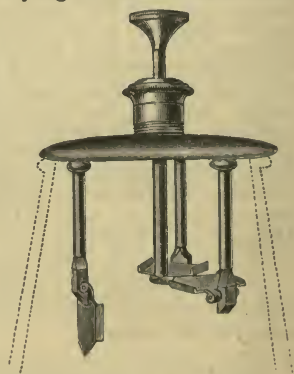


Fig. 57.

and fits in an air-tight manner on the plate of an air-pump. On allowing the two objects to fall when the jar is full of air the heavy one reaches the bottom first. If the air be exhausted and the two then allowed to fall they reach the bottom simultaneously.

66. Acceleration of falling bodies the same for all bodies. We have thus arrived at the important result that in a vacuum all bodies fall to the ground with the same uniform acceleration. This acceleration varies slightly at different places; in England it is approximately 981 cm. per second per second.

In the case of dense bodies the effect of the air is small and we may apply the above results to bodies falling freely through the air.

Hence in a given locality the acceleration with which all heavy bodies fall is the same.

Another and more exact verification of this important law will be given later, see Section 131.

EXPERIMENT 14. *To find a value for* g *the acceleration due to gravity.*

It has been shewn in EXPERIMENT 12 that a falling body moves with uniform acceleration, now in the case of a body so moving we see by putting t equal to unity in the formula $s = \frac{1}{2}at^2$ that *the acceleration is measured by twice the distance which the body describes from rest in the first second*, for we have in that case $a = 2s$ when $t = 1$.

But it has been shewn in EXPERIMENT 11 that a falling body moves over 490·5 cm. or 16·1 feet in the first second of its fall. Therefore we infer that the value of g is $2 \times 490{\cdot}5$ or 981 cm. per second per second. This is equivalent to 32·2 feet per second per second.

67. Weight. A body falls to the earth with uniform acceleration which we denote by g, moreover in a given locality g is constant for all bodies, the mass of the body also is constant, hence the rate of change of the momentum of the body which is measured by mg is constant. But the rate of change of momentum is the **Impressed Force.** Hence the impressed

force is in this case constant. This impressed force is called the **Weight** of the body, if we denote it by W we have the relation $W = Mg$.

Thus in a given locality the weight of a body is constant and in England on the C.G.S. *system it is found in proper units*[1] *by multiplying the mass in grammes by* 981.

Again we have from the above the relation

$$g = \frac{W}{M}.$$

Now in a given locality experiment has shewn that g is the same for all bodies, thus at a given spot the ratio **Weight** to **Mass** is the same for all bodies. *The weight of a body is proportional to its mass;* if we take two lumps of matter one of which has twice the mass of the other the weight of the one will be twice that of the second. We shall see later how this fact is made use of in comparing by weighing the masses of various bodies.

The principle[2] which has just been enunciated that the weight of a body is proportional to its mass is a consequence of the definition of weight and of the experimental fact that in a given locality g is constant for all bodies.

We proceed now to consider some cases of motion in which the moving body is prevented by some means or other from falling freely.

68. Constrained Motion under Gravity.

When a body is allowed to fall freely its velocity soon becomes too great to permit of measurement. Various arrangements have been devised to obviate this. Thus Galileo observed the motion of a ball rolling down a groove in a smooth inclined plane.

He made a series of marks down the groove at distances 1, 4, 9, 16, etc., from the starting point, and found that these marks were passed by the ball at times represented by 1, 2, 3, 4... ; the distance traversed was proportional to the square of the time. Thus the acceleration was constant.

[1] See Section 83. The value of the factor varies slightly in England.

[2] On account of the importance of the principle further and more exact experimental means of verifying it will be given in Section 131.

The same end, the reduction of the velocity to a measurable amount, is attained by the use of

Atwood's Machine.

In this apparatus two equal[1] masses P, Q, Fig. 58, are suspended over a light pulley A by means of a fine string: the pulley is mounted so as to experience very little friction and should be as light as is consistent with strength, the masses should be considerable.

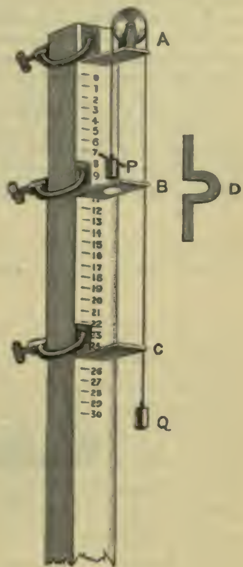


Fig. 58.

This system will remain at rest. A small body called a rider shewn on a larger scale at D in Fig. 58 is placed on the mass P, which commences to descend and the acceleration with which it moves can be observed.

The pulley A is carried on a graduated vertical support ABC. At B is a platform which can be clamped at any position on the support. This platform has a circular hole in its centre, through which the mass P can pass but which is not large enough to allow the passage of the rider. C is a second platform which can be placed so as to stop the motion of P at any point of its fall. When the rider has been stopped by the platform B the system continues to move but the velocity becomes uniform. By observing the motion under these circumstances various consequences can be deduced.

In the following experiments we shall describe the observations which can be most conveniently made by a class of students experimenting with Atwood's machine. These will consist in measuring the intervals

[1] These masses may be supposed to be of the same material; they will then be equal if their volumes are equal, if the rider also be of the same material as P and Q its mass may be compared with that of P by a comparison of volumes; we thus avoid the difficulty of comparing the masses of two bodies of different material.

of time taken by the masses in moving over certain measured distances. Each student or small group of students is furnished with a stop-watch, reading to fifths of a second; at a given signal the demonstrator or one of the students releases the masses, the watches are started simultaneously, and stopped as the rider is removed or as the mass P passes some fixed point on the scale as the case may be.

In some Experiments the procedure may be reversed and the distance measured which the masses describe in a given time. When this is done the pendulum described in Section 65 is often useful; the mass P carrying the rider is supported on a small wooden platform, Fig. 59, hinged in the middle. An iron catch holds this platform in a horizontal position, while

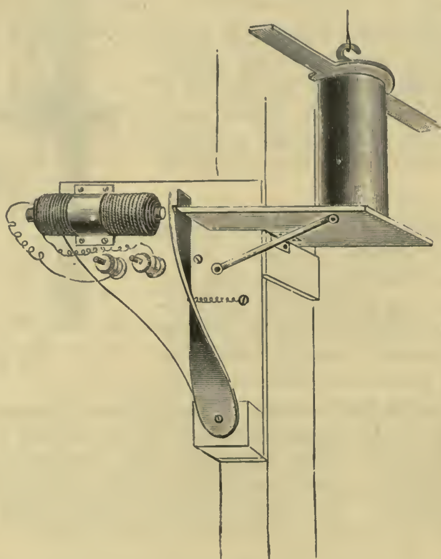


Fig. 59.

a strong elastic band is attached to its under side and pulls it down when the catch is withdrawn. The catch forms the armature of an electromagnet, it is drawn to it each time the magnet is made and is pulled away by a spiral spring when the magnetizing current is broken.

The electric circuit is completed through the spring R of the pendulum (fig. 54) and the connecting screw Q which is connected with

the spring once a second when the pendulum is at the lowest point of its swing. To work the apparatus the pendulum is drawn aside and held fast, the platform is raised and secured by the catch and the rider adjusted. The pendulum is started; as it passes for the first time through its lowest position the current is made and the catch withdrawn with a click; the platform falls and the masses begin to move, the pendulum continues to oscillate and at each second—or half-second if the complete period of the pendulum be one second—a tick is heard as the catch is drawn up to the electromagnet. The platform B can then be adjusted so that the rider is stopped after the motion has continued 1, 2, 3, etc. seconds, and the relation between the distance passed over and the time determined.

EXPERIMENT 15. *To investigate the motion in Atwood's machine*[1] *after the rider has been removed and to shew that in this case* $s = vt$.

Place the ring which catches the rider about 25 cm. below the pulley and make marks on the support at distances of 50, 100, 150 cm. etc. below the ring. Take a stop-watch reading to ·2 of a second, raise the mass which carries the small rider to the top and loose the string. Start the stop-watch as the rider is removed and stop it again as the top of the mass passes the first mark below. This will give the time of passing over the first 50 cm. Repeat the experiment but allow the mass to reach the second mark, 100 cm. from the ring, before stopping the watch. It will be found that this second interval is approximately double of the first. Repeat the experiment using the third mark, and so on. We shall thus prove that the spaces moved through after the rider is removed are proportional to the times of motion. The ratio of the space traversed to the time of traversing it measures the velocity and it is thus seen that the velocity after the rider is removed is constant.

The friction of the pulley will usually be sufficient to produce appreciable error in this result. The effects of the friction may be eliminated by making the mass P slightly greater than Q, adjusting the difference so as to counterbalance the friction and to give to the masses when the rider is removed a uniform velocity.

[1] In Experiments 15—17 the masses P and Q may conveniently be each 1000 grammes and the mass of the rider 10 grammes. A piece of waterproofed fishing line makes the best string.

EXPERIMENT 16. *To shew that the velocity produced up to the instant at which the rider is removed is proportional to the time during which the rider has been on.*

In other words, to prove that the acceleration is constant and that $v = at$.

Place the ring of the Atwood's machine so as to remove the rider after the masses have been moving for some definite time (say 4 seconds) as measured by the stop-watch. Having done this, stop the watch, raise the mass and release it, starting the watch as the rider is removed, and as before measure the time occupied by the mass in descending 100 cm. Now lower the ring until the time taken by the mass in descending from rest to the ring is twice that previously occupied, then observe as above the time taken in falling to a point 100 cm. below the ring. It will be found to be half of that taken in the first case. Thus the velocity in the second case is double of that in the first, and the weight of the rider has been impressed on the masses for twice as long. Hence the velocity produced in a given time is proportional to the time. The ratio therefore of the velocity to the time during which it has been produced is constant and this constant ratio is the acceleration. Thus the formula $v = at$ applies to the velocity generated up to the moment at which the rider is removed.

Thus in Atwood's machine, when the rider is on, the masses move with constant acceleration, after the rider is removed they move with constant velocity.

EXPERIMENT 17. *To verify by means of Atwood's machine the formula* $s = \frac{1}{2}at^2$ *where* s *is the space traversed in* t *seconds by a body moving with constant acceleration* a.

We have just seen that when the rider is on the masses move with constant acceleration. Adjust the ring as in the first part of EXPERIMENT 16 so as to find the space passed over in 4, 6, 8, ... seconds. Measure the spaces in each case from the point at which the mass is released; they will be found to be in the ratio of 16 to 36 to 64 or of 4^2 to 6^2 to 8^2. Thus the spaces are proportional to the squares of the time. Now according to the formula $a = 2s/t^2$, thus the acceleration is given by multiplying the space by 2 and dividing by the square of

the time. Form a table of the values of $2s/t^2$ obtained from each experiment; they will be found to be the same within experimental errors and will be equal to the value of the acceleration given by EXPERIMENT 16.

Thus for uniformly accelerated motion we have $s = \frac{1}{2}at^2$.

With the masses mentioned in the foot-note to Exp. 15, if the pulley is light and the friction small it will be found that the distances traversed in 4, 6, 8 ... seconds will be about 39·2, $39{\cdot}2 \times \frac{9}{4}$ and $39{\cdot}2 \times 4$ cm. respectively, and the acceleration about 4·9 cm. per sec. per sec. Thus in the first part of Experiment 16, the ring B will be about 39·2 cm. below the point of release, while after the rider is removed the masses will move with a velocity of 19·6 cm. per second, and the 100 cm. will be described in about 5·1 seconds. In the second part, the ring B will be about 156·8 cm. below the point of release, and the space of 100 cm. will be described in about 2·5 seconds.

69. Rate of change of Momentum in Experiments with Atwood's Machine. In the above experiments the mass moved and the rider remain unchanged; we find then that the acceleration remains constant. The velocity and the space traversed depend on the time during which the system has been in motion; the acceleration does not, it is uniform throughout. Thus we observe that when the rider is unchanged and the mass moved remains constant then the acceleration is constant. Now let the mass of each of the bodies P and Q be M grammes, let m grammes be the mass of the rider. Then the mass moved is $2M + m$ and the rate of change of momentum therefore is $(2M + m)\,a$.

This then is the **Impressed Force**. If the values of the quantities be substituted in this expression it will be found that the result is equal to mg the weight of the rider. Hence *in Atwood's machine the weight of the rider is the impressed force*, and we have $(2M + m)\,a = mg$.

With the numbers given in Experiment 15,

$$2M + m = 2010 \text{ grammes},$$

$$a = 4{\cdot}9 \text{ cm. per sec. per sec.}$$

Thus $$(2M + m)\,a = 9849;$$

and since $$m = 10, \text{ and } g = 981,\ mg = 9810.$$

Hence $(2M + m)\,a$ or the impressed force as given by the definition is shewn by experiment to be very nearly equal to the weight of the rider.

70. Further experiments with Atwood's Machine. We will now examine the effect of varying the mass moved. Take two masses P_1, Q_1 each half as great as P or Q.

In this case the masses moved are P_1 and Q_1 together with the pulley and the rider. Now the pulley and the rider being both of small mass we may say that the mass moved is very approximately[1] half what it was before.

EXPERIMENT 18. *To shew that in Atwood's machine the acceleration produced by the action of a given rider is inversely proportional to the mass moved.*

Replace the masses P, Q on the Atwood's machine by P_1, Q_1 and determine as in Experiment 17, the acceleration by finding the time required to drop some measured distance. Calculate the acceleration a_1 from the formula $a_1 = 2s/t_1^2$, t_1 being the time taken to traverse a distance s. Then it will be found that a_1 is twice a; by halving the mass moved the acceleration is doubled. Thus for example adjust the platform B so that with the large masses P, Q the motion may continue for 10 seconds before the rider is removed. Repeat the observation using the smaller masses P_1, Q_1. It will be found that the distance is now traversed in about 7 seconds. But the accelerations are inversely proportional to the squares of the times in which a given space is described.

Hence $$\frac{a_1}{a} = \frac{10^2}{7^2} = \frac{100}{49} = 2 \text{ approximately.}$$

If follows therefore from this experiment that if the rider be unchanged the acceleration produced is inversely proportional to the mass moved. The product of the mass and the acceleration is constant.

Thus in Atwood's machine when the weight of the rider is constant the product of the mass moved and the acceleration produced is constant.

We will now consider the effect of varying the weight of the rider.

[1] If we wish we can make it exactly half by making P_1 and Q_1 each slightly less than $\frac{1}{2}P$.

EXPERIMENT 19. *To shew that in Atwood's machine the product of the mass moved and the acceleration produced is proportional to the weight of the rider.*

Take a second rider identical with the first; its weight and its mass are the same as those of the first. On placing the two riders on the mass P the weight of the rider is double as great as previously. The mass moved is slightly greater, being increased by the mass of the rider, we may neglect this and say that the masses are approximately the same as in EXPERIMENT 15[1].

Determine now the acceleration by observing the time taken to traverse some distance. The acceleration will be found to be twice what it was previously.

Thus set the ring B so that the masses may move for 10 seconds before the rider is removed. Repeat the experiment using the two riders; then it will be found that this same distance is traversed in 7 seconds. Hence as in EXPERIMENT 18 the acceleration in the second case is twice that observed in the first case; if three riders be used it will be found that the acceleration is trebled.

Thus by combining this result with that obtained in EXPERIMENT 18 we see that *the product of the mass and the acceleration is proportional to the weight of the rider.*

The weight of the rider may be measured by the product of the mass moved and the acceleration produced.

71. Deductions from Experiments on Falling Bodies. There are three points of importance to be noticed which are common to the above experiments:—(1) in all of them the moving system gains momentum; for as the velocity increases with the time, the mass moved remaining the same, the momentum increases also,—(2) in all of them, so long

[1] If we desire to be more accurate we may allow for the slight change in mass by aid of the result obtained in Experiment 18, or we may use two riders one of which is double the other. In the first part of the experiment place the heavier rider on P_1, the lighter on Q_1, the weight producing motion is the difference between the weights of the riders; in the second part place both riders on P_1, the weight producing motion is the sum of the weights of the two, i.e. three times what it was previously.

as the rider remains unchanged, the rate of change of momentum is constant: (3) when the mass of the rider is varied the rate of change of momentum alters in value, remaining constant for experiments with the new rider. Thus momentum is being transferred to the system and, if we know the rate at which this transference is taking place and the original velocity, we can determine the motion at any time.

Newton realized the importance of this quantity—Ma the rate of change of momentum of a moving body—and called it, as we have done, impressed force. Now we have seen from the Definition of Section 63 that the impressed force remains constant so long as the rate of change of momentum remains unchanged. Thus for example the acceleration g of a falling body of constant mass has been proved to be constant, we infer therefore that the force impressed on the body is constant and is equal to Mg.

Now, according to Newton, this impressed force arises from an attraction between the particles of the Earth and those of the falling body, and it is this attraction which is measured by mg and is said to cause the fall of the body.

We cannot strictly prove the existence of this attraction as a *cause* of the motion, all we are really justified in saying is that all the circumstances of the motion not only in the case of falling bodies but in many other cases are consistent with results deduced from the supposition that there is an attraction between the particles of a body and those of the Earth which is measured by Mg.

Thus to take another example we have seen that in the experiments with Atwood's machine, when the weight of the rider is unchanged, the product of the whole mass moved and the acceleration is constant and equal to the weight of the rider. Hence the weight of the rider is the impressed force, this weight is supposed, as in the case of a body falling freely, to arise from the attraction between the Earth and the rider, and to be the cause of the motion.

Hence starting from Force defined as the rate of change of momentum we are led to the conception of some mutual action between bodies measured by this quantity and changing the motion. We must however remember that when we state that the force acting on a body is F, all we know is that momentum is being transferred to the body at the rate of F units per second.

72. Force and Impulse. We can now consider the connexion between Force and Impulse. The impulse measures the whole amount of momentum transferred to the body in

an interval of time; the force measures the rate at which the momentum is transferred.

If during a time t the rate of transference of momentum is constant and equal to F while the impulse or whole amount transferred is I, then the rate of transference is found by dividing the whole change by the time during which it has occurred.

Hence we have

$$F=\frac{I}{t},$$

or

$$I = Ft.$$

When the force is variable this relation still holds provided that the interval t be so small that we may, for that interval, treat the force as constant.

73. Theoretical Mechanics.

In chapters IV. and V. we have discussed some experiments involving simple cases of motion, we have learnt how masses may be compared and have been led to realize the importance of the ideas of momentum and its rate of change to which the name of force has been given.

We are now about to make a fresh start and consider Dynamics as an abstract Science based on certain laws or axioms which were first clearly enunciated by Newton and are called Newton's Laws of Motion. We shall endeavour in the next chapter to explain these laws and to shew how they may be illustrated by the simple cases of motion already discussed; we then go on to assume them as true always and to deduce their consequences in other cases.

We shall not now discuss the question whether these fundamental principles were stated in their best form by Newton. Our present object is to give a consistent account of the Science of Mechanics as it has been developed from Newton's Laws.

CHAPTER VI.

NEWTON'S LAWS OF MOTION.

74. Galileo's Achievements. Galileo investigated the motion of falling bodies, asking the question, How do heavy bodies fall? He shewed that they move with uniform acceleration, which is in a given locality the same for all bodies; he also determined by experiment the relations given by the formulæ

$$v = at, \quad s = \tfrac{1}{2}at^2.$$

Again, calling the weight of a body, which before his time was recognized by the pressure it produced on the hand or table which supported it, **Force,** Galileo shewed that for falling bodies a force could be measured by the acceleration it produced in a given body.

Newton in his Laws of Motion generalized this idea of force as measured by the rate of change of momentum so as to include all cases of motion.

75. Newton's Definition of Force. This is given in the *Principia* as the fourth definition thus: Vis Impressa est actio in corpus exercita, ad mutandum ejus statum vel quiescendi vel movendi uniformiter in directum.

Impressed force is action exercised on a body so as to change its state of rest or of uniform motion in a straight line.

It should be noticed that this definition does not define the measure of a force; it merely states that action exercised so as to change ("to the changing of") a body's state of rest or uniform motion is Force.

In his further definitions Newton calls attention to the fact that the term force as used in his day was measured in various ways. In the Second Law of Motion he states the meaning with which he uses the term and to which throughout the rest of the *Principia* he adheres. Force is now measured in the manner defined by Newton.

76. Newton's Laws of Motion. The definitions of the *Principia* are followed by three Axioms or Laws of Motion[1]. These are given below, and each will be discussed in turn.

LAW I. *Every body perseveres in its state of rest or of uniform motion in a straight line unless it be compelled to change that state by impressed forces.*

LAW II. *Change of motion is proportional to the impressed force and takes place in the direction in which the force is impressed.*

LAW III. *To every action there is always an opposite and equal reaction, or the mutual actions of two bodies are always equal and opposite.*

77. The First Law of Motion. The first point to notice about this law is that it includes the definition of force. For the definition states that force is action exercised on a body to change its state of rest or motion, whilst, according to the law, the state of rest or motion will not change unless force be exerted. So far the two are the same: the law however states more than this; it defines the state of motion in which a body will persevere unless there is impressed force. If in motion, the body will continue to move with uniform speed in a straight line; if the speed alters or the direction of motion changes, force is said to act on the moving body; if at rest, the body will continue at rest unless

[1] In the original Latin the laws are

Lex I. Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum; nisi quatenus a viribus impressis cogitur statum illum mutare.

Lex II. Mutationem motus proportionalem esse vi motrici impressæ, et fieri secundum lineam rectam qua vis illa imprimitur.

Lex III. Actioni contrariam semper et æqualem esse reactionem; sive corporum duorum actiones in se mutuo semper esse æquales et in partes contrarias dirigi.

acted on by force. Now the action which Newton calls force arises from the presence of matter; and there is reason for believing that every particle of matter in the universe is influenced in its motion by every other particle; when however the distance between the particles is great, the action between them is extremely small. As it is impossible therefore to have a body entirely free from the action of other bodies it is impossible for us to verify by experiment the first law of motion. The law however asserts that if we could completely isolate a body from the influence of all other bodies it would remain at rest or move with constant speed in a straight line,—the motion of a body will not vary unless influenced by other bodies.

This characteristic of bodies is called Inertia, the first law states the principle of **Inertia.**

But while we notice that matter, as inert, is incapable of self-acceleration, incapable that is of changing the speed or direction of its movement, facts justify us in assuming that the motions of any (and every) two particles are mutually affected by each other's presence, the mutual effect in all cases diminishing as the distance between the particles is increased.

Now, though we cannot prove the first law by direct experiment (Newton states it as an Axiom), we can shew that it is consistent with observation; the nearer we approach to the circumstances under which the law is stated to be true the more nearly do observations agree with the results which should follow from the law. A stone will slide further on ice than on a rough road; the friction between the stone and the ice is less than that between the stone and the road, and it is this friction which stops the motion. A lump of iron or lead resting on the ground will not move of itself. Action from some other portion of matter is needed to start it. A body set in motion tends to continue moving.

A ball dropped from the mast-head of a moving vessel strikes the deck at the foot of the mast. The ball at the moment at which it is dropped is moving forwards with the velocity of the ship. This velocity continues during the fall; the ball acquires as well a vertical velocity, it moves with uniform acceleration towards the Earth; but the horizontal velocity remains the same as that of the ship; hence it falls at

the foot of the mast. If a horse suddenly stop there is a tendency for its rider to be pitched forward over its head. The passengers on the back seats of a railway-carriage which is stopped by a collision are thrown forwards against the front of the carriage. A man who steps backwards off an omnibus tends as soon as his feet touch the ground to fall forward on his face; the upper part of his body continues to move with the velocity of the 'bus, his feet are stopped by contact with the ground. In all these cases we have examples of the tendency of motion to continue.

Thus while an appeal to our experience shews us that the law does not contradict observation we cannot thereby prove the law. Our belief in it and in the other laws of motion is really founded on a complicated chain of reasoning. Assuming the laws to be true we can solve the various complicated problems of mechanics; if we find that the solutions which we obtain agree in all cases with observation, and that the agreement is more complete the more completely we apply the laws, we may infer without error that the fundamental principles from which we start are true. Newton applied the laws in combination with his law of gravitation to Astronomy, and shewed how the motions of the planets could be determined, how the eclipses of the Sun might be foretold, and the place from which they would be visible fixed by calculation.

The exact accordance of observation with prediction justifies us in accepting the Laws of Motion as fundamental truths. Mechanics has become a deductive science based on certain definitions and axioms. Experiment and the observation of certain simple cases of motion led Galileo and Newton to recognize certain principles as fundamental. In the Laws of Motion, Newton generalized these principles and applied them to all cases of motion.

78. The Second Law of Motion. *Change of motion is proportional to the impressed force, and takes place in the direction in which the force is impressed.*

In the second definition prefixed to the laws, Newton states that motion, as the term is used in the law, is to be measured *by the product of the mass and the velocity*; it is therefore the

same as the quantity which we have defined as Momentum, while in the eighth definition he explains that Force is measured by the change of Motion (i.e. Momentum), which it produces in *a given time.* Thus by change of motion we are to understand change of momentum occurring in a given time; it is most convenient to take this time as the unit of time, one second; and then *change of motion* as used by Newton becomes in modern language, *Rate of change of Momentum.* We may thus re-state the second law:

LAW II. *Rate of change of Momentum is proportional to the impressed Force and takes place in the direction in which the force is impressed.*

In other words, the law asserts that, when the momentum of a body varies, the rate at which the momentum is changing measures the Impressed Force completely, both in magnitude and direction; this result we have already arrived at in Section 63. Taken in connexion with the definition of Force the law tells us further that we are to take the rate of change of momentum as a measure of action exercised on the body so as to change its motion.

In most cases which occur in Nature the rate of change of momentum of a moving body depends on the position of the body with reference to other bodies. If then we know the position of the body, it is possible to say at what rate its momentum is being changed. Thus if we have two isolated particles free from all external action, each has an acceleration towards the other which is proportional to the mass of the other particle and inversely proportional to the square of the distance between them, the rate of change of momentum is the same for the two; it is proportional to mm'/r^2, where m, m' are the masses of the particles and r their distance apart. This quantity which depends only on the distance between the two particles and on their masses is called the force between them. We can calculate the force impressed on a particle, in this case, without knowing how it is moving or what its velocity is. Assuming the mass of the particle to be constant we are thus given its acceleration or the rate at which its velocity is changing in all positions.

From this, if we know the velocity with which the particle starts, it is possible to determine by mathematical reasoning its path and its velocity at any future instant. We have one simple case of this in the problem of a falling body: the acceleration is constant, the velocity is therefore given by $v=u+at$, the space traversed by the formula $s=ut+\frac{1}{2}at^2$.

The fact that in many cases the rate of change of momentum—the force—is constant or depends only on the position of the body is what gives "Force" its importance in Mechanics. We need not look upon it as some external agent causing the motion, it is sufficient for us to know that for each position of a moving body its acceleration is definite, both in magnitude and direction. It does of course happen that in some cases the acceleration depends on the velocity of the particle, such problems are more difficult to deal with. When we say that the force acting on a certain body is X, all we know is that this body is gaining X units of momentum per second.

79. Measurement of Force. Force like other quantities in Mechanics is measured in terms of a unit of its own kind. The second law does not define this unit, for it merely says that Force is proportional to the rate of change of momentum; it would obviously be convenient if we could say that force is *equal* to the rate of change of momentum, and this statement will afford a definition of the unit force, for suppose the momentum of a body to be changing in each second by unity, then the impressed force is unity. We have thus the following definition of unit force:

DEFINITION. *When the momentum of a body changes in each second by unity the impressed force is the* **Unit of Force.**

Hence when the change in momentum per second is 2 units of momentum the impressed force is 2 units of force, and when it is stated that the impressed force is F it is implied that the momentum increases by F units per second.

PROPOSITION 21. *To obtain an equation connecting the rate of change of momentum of a body and the impressed force.*

Let u be the initial velocity of a particle of mass m, v its velocity at the end of t seconds, a its acceleration, and F the impressed force. We suppose F to be constant during the time.

Then in one second the particle gains F units of momentum. But the original momentum was mu, the momentum after t'' is mv.

Hence change of momentum in t seconds is $mv - mu$. Therefore change of momentum in 1 second is $(mv - mu)/t$.

Therefore $$F = \frac{mv - mu}{t} = \frac{m(v - u)}{t}.$$

But $$v = u + at.$$

Hence $$\frac{v - u}{t} = a.$$

Therefore $$F = ma.$$

Thus the product of the mass and the acceleration is equal to the impressed force provided that F, m and a are measured in a consistent system of units in which the impressed force is unity, when the gain of momentum per second is unity.

80. The C.G.S. Unit of Force. According to the C.G.S. system the unit of mass is 1 gramme, the unit of velocity is a velocity of 1 centimetre per second. Hence if the velocity of a mass of 1 gramme increases per second by 1 cm. per second the mass gains momentum at unit rate. The impressed force therefore is the C.G.S. unit of force. The C.G.S. unit of force is called a Dyne.

DEFINITION OF ONE DYNE. *When a mass of one gramme gains per second a velocity of one centimetre per second, the impressed force is one* Dyne.

Hence if a mass of m grammes has an acceleration of a centimetres per second per second, the impressed force is ma Dynes.

Or again, if we know that the impressed force is F dynes, and the mass m grammes, then the acceleration a in centimetres per second per second, is given by the equation

$$F = ma,$$

or $$a = \frac{F}{m}.$$

81. The F.P.S. Unit Force. On the English or F.P.S. system of units the unit mass is 1 pound; the unit of

velocity is a velocity of 1 foot per second. Thus if the velocity of a mass of 1 pound increases per second by a velocity of 1 foot per second, the impressed force is the English or F. P. S. unit force. The F.P.S. unit force is called the Poundal.

DEFINITION OF ONE POUNDAL. *When a mass of one pound gains per second a velocity of one foot per second, the impressed force is one* **Poundal.**

Hence if a mass of m pounds has an acceleration of a feet per second per second, the impressed force is ma **Poundals.**

Examples. **(1)**. *The velocity of a mass of* 10 *grammes is changed in* 5 *seconds from* 25 *to* 125 *cm. per second, find the force.*

The change in velocity in 5 seconds is

$$125 - 25 \text{ or } 100 \text{ cm. per second;}$$

$\therefore$ the acceleration or change in velocity per second is 100/5 or 20 cm. per sec. per sec.

The mass is 10 grammes.

Hence the force is 10×20 or 200 dynes.

(2). *A train whose mass is* 20 *tons moves at the rate of* 60 *miles an hour; after steam is shut off it is brought to rest by the brakes in* 500 *yards. Find the force exerted, assuming it to be uniform.*

[To solve this problem we must express the velocity in feet per second, the mass in lb., then find the retardation and hence the force.]

A velocity of 60 miles an hour is $\frac{1760 \times 3 \times 60}{60 \times 60}$, or 88 feet per second.

Let a be the retardation; this velocity is destroyed in a space of 500 yds.

Hence applying the formula $v^2 = 2as$, we have

$$2 \times a \times 500 \times 3 = 88^2 = 7744;$$

$$\therefore a = 2{\cdot}5813 \text{ feet per sec. per sec.}$$

Now the mass of the train is

$$20 \times 20 \times 112 \text{ or } 44800 \text{ pounds.}$$

Hence the

$$\text{Force} = 44800 \times 2{\cdot}5813 \text{ poundals}$$
$$= 115642 \text{ poundals.}$$

(3). *Find the force if in* 1 *second a mass of* 1 *gramme gains a velocity of* 981 *cm. per second.*

The acceleration is 981 cm. per sec. per sec., and the mass is 1 gramme. Thus the force is 1×981 or 981 dynes.

(4). *How many dynes are there in a poundal?*

A poundal is the impressed force when a mass of 1 lb. has an acceleration of 1 ft. per second per second.

Now 1 foot contains 30·48 cm., and 1 pound contains 453·6 grammes.

Therefore if the impressed force be 1 poundal a mass of 453·6 grammes has an acceleration of 30·48 cm. per sec. per sec.

But the number of dynes equivalent to this is $453 \cdot 6 \times 30 \cdot 48$ or 13826 —omitting decimals;

$$\therefore 1 \text{ Poundal} = 13826 \text{ Dynes.}$$

Thus we see that a Dyne is a very small force compared with a Poundal.

82. Comparison of Forces. Forces are measured by the momenta which they communicate per second to any mass. If we restrict ourselves to one body the momentum gained per second by that body will be proportional to its acceleration. Two forces then can be compared by comparing the accelerations communicated by them to the same body.

Thus, for example, place a rider on one of the suspended masses on an Atwood's machine and observe the acceleration; change the rider and again observe the acceleration; the weights of the two riders are proportional to the two accelerations [1].

Or, again, it is found by experiment that near the Equator a body falls with an acceleration of about 978 cm. per sec. per sec., while in high latitudes near the Pole the value of this acceleration is about 983 cm. per sec. per sec. Thus the weight of a body is greater near the Pole than near the Equator in the ratio of 983 to 978; in going from the Equator to the Pole a body gains in weight about 5 parts in 1000. (See Section 134). Observations on the Moon shew that it has an acceleration towards the Earth of about ·27 cm. per second per second. The acceleration towards the Earth of any body when close to the Earth is about 980 cm. per second per second. Thus if the Moon were close to the Earth its weight would be increased in the ratio of 980 to ·27 or about 3600 times.

[1] It is assumed here that the mass of the rider is small compared with the suspended masses.

DEFINITION OF EQUAL FORCES. *Two forces are said to be equal when the velocities which they communicate per second to the same mass are equal.*

83. Comparison of Masses. We have seen already (§ 51) how masses may be compared by Prof. Hicks' ballistic balance, and from experiments with it we have obtained an idea of the meaning of the term mass.

The second law of motion gives us another method of comparing masses consistent with the above.

For let M_1, M_2 be two masses, a_1, a_2 the accelerations communicated to them by a given force F. Then by the second law

$$M_1 a_1 = F = M_2 a_2.$$

Thus
$$\frac{M_1}{M_2} = \frac{a_2}{a_1}.$$

Thus two masses are inversely proportional to the velocities which are communicated to them per second by the same force.

DEFINITION OF EQUAL MASSES. *Two masses are equal when a given force communicates*[1] *to them per second the same velocity.*

The simplest method theoretically in which we could apply this method of comparing Masses would be to imagine the two masses free from external action and capable of motion under their mutual action only; the accelerations of each body ought then to be observed and would be proportional to the mass of the other body. But in practice any such method is impossible. Now, we have learnt from experiments with Atwood's machine that in a given locality the weight of a body is a constant force. We may use, then, the following method for comparing masses. Place two equal masses on an Atwood's machine and observe the acceleration communicated by a given rider. Replace these masses by two other equal masses and again

[1] We may put this otherwise thus. When two bodies are gaining momentum at the same rate the force acting on each of them is the same. If they are also gaining velocity at the same rate the masses of the two are equal.

observe the acceleration. The masses suspended in the two cases are inversely proportional to the accelerations.

We may sum up the conclusions of the last two sections thus:

By considering the accelerations communicated to the *same mass by different forces*, we see that

(1) Two forces are equal when they communicate the same acceleration to a given mass.

(2) A force is proportional to the acceleration it communicates to a given mass.

Whilst by considering the acceleration communicated to *different masses* by the *same force*, we see that

(1) Two masses are equal when a given force communicates the same acceleration to each.

(2) The mass of a body is inversely proportional to the acceleration communicated to it by a given force.

84. Falling Bodies and the Second Law of Motion. When a body falls freely it moves downwards with acceleration g; let M be its mass, the impressed force is the weight of the body, let it be W. Then since W is the force which communicates to the mass M its acceleration g we have the relation

$$W = Mg.$$

The acceleration g is constant for all bodies in a given locality, but varies from place to place.

In obtaining this result we have assumed that our measurements are made in a consistent system of units. The force must be measured in units such that the unit force communicates per unit of time to the unit of mass unit velocity; if we work on the C.G.S. system the force is measured in Dynes. If we work on the F.P.S. system it is measured in Poundals.

85. Value of a Dyne. We can use this result to enable us to specify in a more concrete form what a force of 1 dyne is.

For consider a mass of 1 gramme so that M is 1 in the formula and W stands for the weight of a mass of 1 gramme.

We have

$$W = g,$$

or *the weight of a mass of* 1 *gramme contains* g *dynes.*

Now, as we have said, the value of g varies at different parts of the Earth. In England we may take it as equal to 981 cm. per sec. per sec. We learn therefore

(1) That the weight of a mass of 1 gramme varies at different parts of the Earth.

(2) That the weight of a mass of 1 gramme in England contains 981 dynes.

Thus 1 dyne is $\frac{1}{981}$ of the weight of 1 gramme in England.

Or, in other words, divide a mass of 1 gramme into 981 parts. The weight of each part in England is 1 dyne. Since 981 is not very different from 1000 we may say that roughly a dyne is one-thousandth part of the weight of 1 gramme, or is equal to the weight of a milligramme, so that if we apply to a mass of 1 gramme a force equal to the weight of 1 milligramme it will acquire approximately an acceleration of 1 cm. per sec. per sec. (The real value of the acceleration will be ·981 cm. per sec. per sec.)

86. Value of a Poundal. The equation $W = Mg$ will apply equally well to the F.P.S. system. In this case M is in lb., W in poundals, g in feet per sec. per sec., and we may take approximately $g = 32$ feet per sec. per sec.

Consider now the case in which the mass is 1 lb. Then $M = 1$, and W is the weight of a mass of 1 lb.

Thus $W = g = 32$ approximately, and we have the result that

The weight of a mass of 1 pound contains 32 poundals, or

1 Poundal = $\frac{1}{32}$ of the weight of 1 lb. = the weight of half an ounce.

Thus if we apply a force equal to the weight of half an ounce to the mass of 1 pound the mass will acquire approximately an acceleration of 1 foot per second per second. (The result is only approximate, because g is not accurately equal to 32 feet per sec. per sec.)

87. Relation between Weight and Mass. Since at any point on the Earth's surface the acceleration of a falling body is the same for all bodies, it follows that the weights of two bodies are proportional to their masses, for the ratio of the weight to the mass measures in each case the acceleration produced, and this is the same for the two. Thus at any given point on the Earth the weight of a body is proportional to its mass.

Since the acceleration of a falling body is different at different points on the Earth the ratio of the weight of a given body to the mass of that body differs from point to point; the mass of the body is the same everywhere, hence we infer that the weight of a body differs from point to point. The mass of a body is an invariable quantity; the weight of a body depends on its position.

CHAPTER VII.

FORCE AND MOTION.

88. The Action of Force. In Chapter v. Force has been defined as Rate of Change of Momentum. We do not need for the purposes of Mechanics to discuss the question whether there is some cause external to a moving body which acts upon it and makes it move; we can leave the question of efficient causes out of sight. We can observe the velocity and the acceleration of moving bodies, we find in many cases that the product of the mass and the acceleration does not depend on the motion of the body, but is either constant or depends on its position relative to surrounding bodies. This it is true lends plausibility to the idea that this quantity—the impressed force as it has been called—is something external to the body efficient in making it move; thus the phrases "the forces *acting* on the body," "the forces *producing* motion," and the like are in common use.

We have however no right to say that force *produces* motion, or that force *acts* on a body, if we attach to the words 'produce' and 'act' their ordinary meaning, implying the existence of some agent or cause to which the motion can be assigned and define force as above. At the same time we may conveniently use the phrase "the forces acting on the particle" and the like if we do it in a sense limited by our own definition. *All that we mean by the statement that a force is acting on a body is that the momentum of that body is changing.* We define the *acting* of force thus :

DEFINITION. *When the momentum of a body is changing gradually, force is said to* act *on the body.*

Again, a body may be in equilibrium under the action of several forces; in this case the accelerations which the body would have were each force to act singly are so related that their resultant is zero.

The relation between acceleration and the corresponding force is always given by the equation,

$$F = ma.$$

89. Law of Gravitation. The change of motion of any particle depends on its position with regard to other bodies.

Each particle in the universe has an acceleration towards all the other particles; the acceleration which we observe is the resultant of these innumerable component accelerations.

If we have two particles A and A_1 of masses m and m_1 at a distance r apart, A has, as we have already said, an acceleration towards A_1, and its amount is m_1/r^2, A_1 has an acceleration towards A, and its amount is m/r^2; the impressed force on A is therefore mm_1/r^2 towards A_1, while that on A_1 is m_1m/r^2 in the direction A_1A. These two forces are equal and opposite; we may express this fact by saying that there is an attraction mm_1/r^2 between the particles A and A_1.

Newton, in the law of gravitation, asserts that this is true for every pair of particles in the Universe. In the *Principia* he calculated the Motion of the Planets and their Satellites, assuming this law to be true; the fact that motion so calculated agrees with observation justifies his assumption.

The law is usually stated thus:

LAW OF GRAVITATION. *Every particle of matter attracts every other particle with a force which is proportional to the product of their masses and inversely proportional to the square of the distance between them.*

We can put the law rather differently thus, and in this form it represents more accurately the results of observation: *Every particle in the Universe has an acceleration towards every other particle. The amount of the acceleration towards any*

second particle is proportional to the mass of this second particle, and inversely proportional to the square of the distance between the two.

To determine then the acceleration of a particle completely we calculate the resultant of these component accelerations.

Thus any particle near the Earth has accelerations directed to the particles of which the Earth is composed, and also towards those of the Sun, the Moon and the stars. Owing however to the immense distances of these bodies these latter accelerations may be neglected and the effect of the Earth only calculated.

Now the Earth is very nearly a sphere, and Newton shewed that for an external particle the resultant of the acceleration due to the particles of a homogeneous sphere is the same as it would be if the sphere were replaced by a single particle at its centre; the mass of this particle being equal to that of the sphere. It is easy thus to calculate the acceleration of a particle free to move near the Earth; if M be the mass of the Earth, and R its radius, the value of this acceleration will be that due to a mass M at a distance R; its amount therefore is M/R^2, and its direction will be towards the centre of the Earth.

Thus a particle close[1] to the Earth's surface will have a vertical acceleration M/R^2, and this is the same whatever be the mass of the particle.

The impressed force on the particle will therefore be mM/R^2, and this is w the weight of the particle.

Thus we have

$$w = \frac{mM}{R^2}.$$

Again, we may look upon a body of finite volume as an aggregation of particles. Each of these has the same acceleration M/R^2 towards the Earth; thus the whole body has this

[1] We assume that the particle is so near the Earth's surface that its height h may be neglected compared with R: the true value of the acceleration is $M/(R+h)^2$, and if h is small compared with R (4000 miles) we may call the acceleration M/R^2.

acceleration, and if m now represent the mass of the whole body, and W its weight, we have

$$W = \frac{mM}{R^2}.$$

Thus the weight of a body is proportional to its mass; the acceleration with which it falls is equal to the mass of the Earth divided by the square of its radius and does not at all depend on the body.

On the other hand, the Earth is not accurately a sphere, its polar diameter is less than its equatorial diameter, and though we may, without serious error, calculate the acceleration of the particle as though the Earth were spherical, and suppose the whole mass concentrated at the centre we must remember that the quantity R, the distance between the Earth's centre and the point for which our calculations are made, differs for different points on the Earth, being less near the poles than near the equator.

For this reason the acceleration is greater near the poles than near the equator.

The motion of the Earth round its axis tends also to reduce the acceleration as the particle approaches the equator (see § 143).

Thus Newton's law of gravitation leads to conclusions in accordance with those deduced from the second law of motion as to the relation between mass and weight. For a definite position on the Earth the weights of all bodies are proportional to their respective masses; at different points on the Earth the same body has different weights.

In the foregoing section we have stated that the acceleration of a mass m' when at a distance of r centimetres from a second mass m is m/r^2, and that the force is mm'/r^2. But the acceleration so measured is not given in centimetres per second per second; if it were, then if we placed two small bodies each 1 gramme in mass at a distance of 1 centimetre apart, they would have an acceleration toward each other of $1/1^2$ or 1 centimetre per second per second.

Now the acceleration of two masses under such circumstances has been determined by means of the torsion balance and in other ways, and

has been found to be very much smaller than 1 cm. per sec. per sec. It is in fact about

$$6 \cdot 698/10^8 \text{ cm. per sec. per sec.}$$

and the impressed force on each mass is

$$6 \cdot 698/10^8 \text{ dynes.}$$

Hence the impressed force on a particle m grammes in mass at a distance of r centimetres from a particle of equal mass is

$$\frac{6 \cdot 698}{10^8} \frac{m . m'}{r^2} \text{ dynes.}$$

90. Gravitational Unit of Force. We can if we like measure all our forces in terms of the weight of some given body, say 1 lb. In such a case of course when speaking of a force P we do not mean a force of P dynes, or P poundals, but a force P times as great as the weight of 1 pound. Such a unit is known as a **gravitational unit of force.** It depends on the attraction between the Earth and a body having a mass of 1 pound. Now this attraction depends on the position of that mass on the Earth; it is greater as said above near the poles than near the equator; thus the gravitational unit of force is different at different points, a force, 10 say, would be really a larger force near the poles than near the equator; it would mean in each case ten times the weight of a certain lump of matter and this force is greater in high latitudes than in low.

If we are working with gravitational units we cannot use the equation $F = ma$, for this supposes that the unit force communicates unit acceleration to the unit of mass; now in gravitational units this supposition is not true, the unit force is the weight of 1 pound, the unit of mass is the mass of 1 lb. and the unit force communicates acceleration g to the unit of mass; the weight of 1 pound contains g poundals, the gravitational unit is g times as great as the absolute unit (§ 79).

We can determine the relation between the acceleration and the force when gravitational units are employed thus.

Let W be the weight of the body in pounds, F the force acting on it in pounds' weight, a the acceleration produced in feet per second per second.

Then we have

W communicates to the given body an acceleration g,

F .... a.

But two forces are respectively proportional to the accelerations they communicate to a given body.

Hence $$F : W = a : g.$$

$$\therefore \frac{F}{W} = \frac{a}{g}.$$

$$\therefore F = \frac{W}{g} . a.$$

Example. (1) Find in gravitational units the force required to give in $\frac{1}{5}$ of a second to a mass of 1 cwt. a velocity of 100 feet per second.

Since in $\frac{1}{5}$ second a velocity of 100 ft. per second is produced, a velocity of 500 feet per second is produced in 1 second.

Hence the acceleration is 500 ft. per sec. per sec.

The weight of 1 cwt. is 112 lb. wt.; thus taking g as 32

$$F = \frac{112}{32} . 500 = 1750 \text{ lb. weight.}$$

Thus at a place at which g is equal to 32, a force of 1750 lb. weight acting on 1 cwt. will in $\frac{1}{5}$ of a second produce a velocity of 100 feet per second.

(2) Find what velocity this force will in $\frac{1}{5}$ of a second produce in a mass of 1 cwt. at a place at which the value of g is 32·2 ft. per sec. per sec.

Let a be the acceleration.

Then $$\frac{a}{32 \cdot 2} = \frac{1750}{112} = \frac{500}{32};$$

$$\therefore a = \frac{32 \cdot 2}{32} \times 500 = 503 \cdot 125 \text{ ft. per sec. per sec.}$$

Hence the velocity produced in $\frac{1}{5}$ sec. is 100·625 feet per second.

Hence when gravitational units are used two forces *nominally* the same produce different effects at different points on the Earth.

91. Equilibrium. When a body is in equilibrium it has no acceleration; the total impressed force therefore is zero; now it is often desirable to look upon this state of no acceleration as the consequence of the superposition of two or more accelerations which are so related that their resultant is zero. When there are only two such accelerations they are clearly equal and opposite.

Thus consider a body supported by a string: were it free it would move to the Earth with uniform acceleration g; its freedom however is modified by its connexion to the string; since it

has no acceleration the action of the string must be such that were the Earth removed the body would begin to move upwards with acceleration g; in this case the impressed force would be upwards and equal to mg. This impressed force is called the tension of the string. The particle is at rest, hence the upward tension of the string is equal to the weight of the particle and the total impressed force is zero.

Again, consider a body attached to a vertical spiral spring, let the body be at first supported in such a position that the spring is unstretched, gradually lower the support, the body falls, and the spring is stretched until the body is left suspended from the spring. In this case also there is action between the spring and the body. There is an impressed force on the spring equal to the weight of the body.

Just then in the same way as we may look upon the actual acceleration of a body as the resultant of a number of component accelerations, so we may consider the impressed force as the resultant of a number of impressed forces. We shall see shortly how the component forces and their resultant are related together.

92. Comparison of Masses by "Weighing."

We have seen that it follows from the definition of force that the weight of a body is proportional to its mass; this result also is in accordance with Newton's law of gravitation. Two bodies then of equal weight are equal in mass; this fact may be made use of as a means for comparing masses. This is the principle of the ordinary balance (see *Statics*, Section 59); the balance enables us to determine when the weights of two bodies are equal. When this is the case we infer that the masses are equal also.

It is this equality of mass which we usually wish to secure in weighing. In buying a pound of tea or a pound of sugar the customer cares nothing about the attraction of the Earth for the tea or sugar. He wishes to know that he is obtaining for his money an amount of tea equal to that which he has been in the habit of receiving for that sum.

This end is secured most readily by comparing in each case the mass of tea purchased with some standard mass, a pound or kilogramme. The comparison by the balance of the weight of the tea and of the standard

pound affords the easiest method of comparing their masses. If he buys two equal masses of tea of the same quality for the same sum of money he infers that the price he is paying per cup for tea of a definite strength remains unchanged, and this is what he wishes to know.

The fact that the same term—pound or kilogramme as the case may be—is used in ordinary language both for mass and weight, is no doubt productive of confusion. A pound is strictly a denomination of mass but it and other names of mass are also used as a denomination of force; thus Engineers speak of a pressure of so many pounds to the square inch or of the breaking stress of a piece of material being so many tons; in such cases the phrase is an abbreviation. A force of P pounds means a force equal to the weight of a mass containing P pounds.

93. Methods of measuring Force. The method of measuring force by the acceleration it communicates to a body is not always the most convenient. For many purposes as we have seen a force may be measured in terms of a weight; a force of a certain amount F acts on a body in a given direction, we can imagine a string attached to the body and passing from it in the given direction over a smooth pulley. Hang a mass whose weight is equal to the force on to the string, then the force acting along the string is represented by the weight.

If the force acts in a vertical direction on a body it may be represented by the weight of a mass placed directly on the body.

Now in some cases force when it acts on a body changes visibly the size or shape of the body. If a body of considerable mass be placed on an indiarubber ball, the ball is squeezed and flattened; if the same mass be suspended by a piece of indiarubber the indiarubber is lengthened; in these cases force acting on the body visibly[1] alters its shape. In many such cases we can shew that the change in shape is proportional to the force.

Thus take a spiral spring[2], fasten one end to a fixed support and suspend a light scale-pan from the other end.

[1] The shape or size of nearly all bodies is altered by the action of force; with many bodies however the alteration is too small to be detected unless special means of observation are employed.

[2] Such a spring is easily made by winding a piece of steel or brass wire about 1 mm. in diameter in a spiral coil on a rod of circular section

Attach a piece of wire to form a horizontal pointer to the lower end of the spiral spring and adjust a scale in a vertical position as shewn in Fig. 60, so as to mark the position of the pointer. Note the position of the pointer on the scale, then place a mass in the scale-pan, the spring is extended. Note the position of the pointer and remove the mass; unless the mass be too large[1] the pointer will be found to return to its original position.

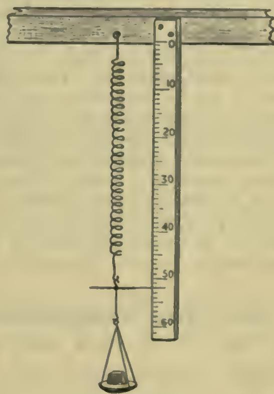


Fig. 60.

On replacing the mass, the spring is again extended and to the same amount as before. When the body is placed in the scale-pan a force, the weight of the body, acts on the scale-pan, the spring is stretched and for a given force the extension is constant. Now replace the body in the scale-pan by one of double the mass and therefore of double the weight. Observe in this case the position of the pointer and again measure the extension. Unless the weight is too large for the spring it will be found that in this case the extension is double that observed previously. The force acting is doubled and the extension also. By varying the mass we may shew that in all cases, so long as we keep within the elastic limits of the spring, the extension is proportional to the force. Thus the extension of the spring may be made use of to measure the force. This is done in the ordinary form of spring balance such as is used

some 2 or 3 cm. in diameter, the rod is placed in a lathe and turned slowly by hand while the wire is wound on to it under considerable tension.

[1] By loading the scale-pan too heavily the spring may be permanently stretched so that when the body is removed the pointer does not come back to its original position: if this is done the spring is said to be stretched beyond its elastic limits and the relation between extension and force no longer holds.

for weighing letters. The balance is graduated by hanging on masses of 1, 2, 3 etc. pounds. When a body of unknown mass is suspended its mass is determined by observing the position on the scale at which the pointer rests.

It should be noticed that a spring balance measures the force applied. If used to determine mass it will only do so correctly in the latitude at which it was graduated. For suppose it graduated in London, suspend a mass of say 1 kilogramme and carry the whole northwards; the mass suspended remains the same, but as the pole is approached the weight of that mass increases, the force acting on the spring becomes greater and the spring is stretched further, the balance therefore reads over 1 kilogramme, but if it is inferred from this that the suspended mass is greater than a kilogramme the inference is wrong. A similar result though in the opposite direction will take place if the balance be carried towards the equator. Thus the extension of a spiral spring affords another method of estimating force.

94. The Composition of Forces. The motion or the equilibrium of a body depends on its relations to other bodies.

Let us suppose that we know the motion which would follow were the body free in turn from the action of all but one of the bodies which can affect its motion. We know then the forces which act separately on the body, and we wish to deduce from this knowledge what will happen when those forces are combined and act simultaneously.

Now the second law of motion states that rate of change of momentum is proportional to the impressed force and *takes place in the direction of that force.* We can extend the application of this law to the case of a number of forces thus. Calculate the acceleration of the body under the action of each force separately, combine the accelerations according to the parallelogram law, then the resultant acceleration is that which the body will have when the combined forces are impressed on it. The acceleration corresponding to a given force is independent of other velocities or accelerations which the body may possess. The action of each force is unimpeded by the others, the observed motion—or rest—is the result of all.

We shall see however in the following sections (§ 96 seq.) that there is a rule by which a number of forces may be com-

bined and their resultant found. This rule was given by Newton as a Corollary to the laws of motion; it is often simpler to proceed by its aid and then to calculate the acceleration under the action of this resultant force rather than to reverse the process and after finding the acceleration corresponding to each force combine these so as to obtain the actual motion.

Thus the second law of motion in the form in which it has been stated involves the principle that,

The effect of a force on a body, as measured by the rate of gain of momentum, is independent of other forces which may be impressed.

This is sometimes spoken of as the independence of forces.

95. Representation of a Force.

Proposition 22. *To prove that forces can be represented by straight lines.*

To define a force we need to know the number of units of force it contains, the direction in which it acts and the point at which it is impressed. These can be represented by a straight line, for a straight line can be drawn from a given point—the point of application—in a given direction—the line of action of the force—and so as to contain a given number of units of length—the number of units of force in the given force. More briefly the proof can be put thus: A force is measured by the acceleration it can communicate to unit mass, and acceleration can be represented by a straight line.

Thus if we suppose that a length of 1 centimetre represents the unit of force then a line AB, 5 cm. long, drawn from A to B, represents a force acting at A in the direction AB and containing 5 units of force.

96. The Parallelogram of Forces.

Proposition 23. *If two forces impressed on a particle be represented in direction and magnitude by two adjacent sides of a parallelogram the resultant of the forces is represented by the diagonal of the parallelogram which passes through their point of intersection.*

Let OA, OB, Fig. 61, represent the two forces.

Complete the parallelogram $AOBC$ and draw the diagonal OC.

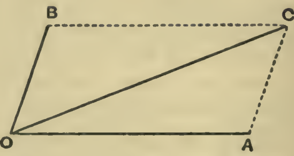


Fig. 61.

A force is measured by the velocity it communicates per second to a particle of unit mass.

Therefore OA, OB represent the velocities which the force would communicate per second to a particle of unit mass.

Therefore, by the parallelogram of velocities, OC would be the resultant velocity of the particle if the forces acted on it for one second. Thus OC represents the force which acting on the particle for a second would communicate to it its actual velocity.

Hence OC represents the resultant force; now OC is the diagonal of a parallelogram whose sides OA, OB represent the forces P, Q respectively. Thus the Proposition is true.

The formulæ and propositions established in §§ 29—32 about the composition and resolution of velocities and displacements will therefore apply to forces. The development of these formulæ applied to bodies at rest gives us the Science of Statics, which is considered with experiments in the second part of this book. The following sections illustrate the resolution of forces and the application of the second law of motion to some simple problems.

97. Problems on Motion.

PROPOSITION 24. *To determine the motion of a body sliding down a smooth*[1] *inclined plane.*

Let m be the mass of the particle, α the angle between the plane and the horizon, R the force between the plane and the particle, a the acceleration of the particle along the plane.

[1] A surface is said to be "smooth" when the direction of the force between it and any body in contact with it is at right angles to the surface.

Resolve the forces along and perpendicular to the plane ABC, Fig. 62, and equate in each case the rate of change of momentum to the force in its direction.

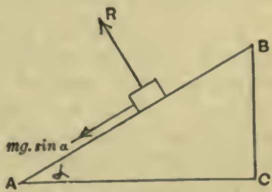


Fig. 62.

The weight of the particle is mg dynes and it acts vertically downwards; this can be resolved into

$$mg \cos \alpha$$

perpendicular to the plane and

$$mg \sin \alpha$$

along the plane. Thus the force perpendicular to the plane is

$$R - mg \cos \alpha.$$

The force along the plane is

$$mg \sin \alpha.$$

There is no acceleration perpendicular to the plane while the acceleration down the plane is a.

Hence

$$0 = R - mg \cos \alpha,$$

$$ma = mg \sin \alpha,$$

$$\therefore\ R = mg \cos \alpha,$$

$$a = g \sin \alpha.$$

Thus the force on the plane is $mg \cos \alpha$ dynes while the particle slides down with uniform acceleration $g \sin \alpha$.

Hence, if l be the length of the plane, t the time taken by the particle in sliding down it, and v the velocity of the particle at the bottom, assuming it to start from rest at the top, then

$$l = \tfrac{1}{2} g \sin \alpha \,.\, t^2,$$

$$v = g \sin \alpha \,.\, t,$$

$$v^2 = 2g \sin \alpha \,.\, l.$$

PROPOSITION 25. *To determine the motion of two particles suspended by a fine string over a smooth pulley.*

Let m and m' be the masses of the particles A and B, Fig. 63, and let m be greater than m', we suppose the mass of the pulley may be neglected. Since m is greater than m', the mass A moves downwards while B moves up; let a be the common acceleration.

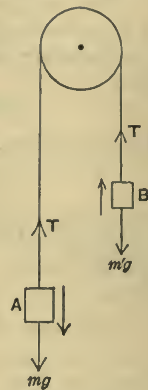


Fig. 63.

The weight of A is mg, that of B is $m'g$, both these forces act downwards but the former acts on A in the direction of its motion, the latter acts on B in a direction opposite to that of its motion. Thus the impressed force in the direction of motion is $(m - m')\,g$.

The mass moved is $m + m'$, hence the rate at which momentum is gained by the system is $(m + m')\,a$.

Therefore equating this to the force

$$(m + m')\,a = (m - m')\,g.$$

Hence $$a = \frac{m - m'}{m + m'}\,g.$$

Thus the system moves with a uniform acceleration which is a definite fraction of that due to gravity. The space passed over and the velocity generated in a given time can be found in the usual way.

PROPOSITION 26. *To find the tension of the string joining the two masses suspended over a pulley as in the last Proposition.*

To obtain this we must consider the motion of each mass separately, remembering that since they are connected by the string the upward acceleration of m' and the downward acceleration of m must be the same. Moreover the tension of the string is the same throughout. Let it be T dynes, then a force T acts upwards on both m and m'. Let the acceleration be a.

The weights of m and m' are mg and $m'g$ dynes respectively and act downwards, and we know by the second law that the

product of the mass and the acceleration is equal to the force in the direction of motion.

Hence for the downward motion of m

$$ma = mg - T,$$

while for the upward motion of m'

$$m'a = T - m'g.$$

By adding the two equations we have as in Proposition 25 above

$$(m + m')\, a = (m - m')\, g,$$

$$a = \frac{m - m'}{m + m'}\, g.$$

Multiply the first equation by m', the second by m and subtract, we then find

$$0 = 2mm'g - T\,(m + m').$$

Hence $$T = \frac{2mm'g}{m + m'}.$$

We have thus found the tension of the string.

The three preceding propositions give us examples of the method to be followed in solving all mechanical problems. We divide the process into three parts: (1) the formation of the equations of motion, (2) the solution of those equations, (3) the physical interpretation of the solution.

Under (1) we express the forces and the rates of change of momentum in terms of symbols and form the equations by equating in accordance with Newton's second law the forces and the corresponding rates of change of momentum; this constitutes the fundamental dynamical part. Under (2) we apply the methods of Algebra or of Trigonometry to the solution of the equations, and obtain the unknown accelerations or forces in terms of known quantities. Under (3) we interpret the solution we have found.

98. Experiments on the value of g. The last propositions furnish us with the theory of experiments which may be used to find g.

EXPERIMENT 20. *To find g by observation on a body sliding down a smooth inclined plane.*

We have seen that if a be the angle of the plane, the acceleration is $g \sin a$ down the plane, and if the time t of moving down a length l of the plane be observed,

$$l = \tfrac{1}{2} g \sin a \,.\, t^2.$$

Hence $$g = \frac{2l}{t^2 \sin \alpha}.$$

Again if h be the height of the plane, l its length, then

$$\sin \alpha = h/l,$$

and $$g = \frac{2l^2}{ht^2}.$$

To make the observation, obtain a sheet of glass or a smooth board with a groove down it about a metre long. Raise one end until it is some 2 or 3 cm. in height above the other, measure this height carefully and also the length of the plane. Place a smooth marble or small ball at the top and observe with a stop-watch the time taken by it in rolling down, then the values of l, h and t are known, and g can be calculated. The results however will not be very accurate because of friction and the difficulty of observing the time accurately.

EXPERIMENT 21. *To determine* g *by means of Atwood's Machine.*

In Atwood's Machine let M grammes be the mass of each of the two large weights, m grammes that of the rider; then omitting the effect due to the mass of the pulley and to friction, the mass moved is $2M + m$, the force producing motion is the weight of the rider or mg dynes. Let the acceleration be a, then the rate of change of momentum is $(2M + m)a$ and we have $$(2M + m)a = mg.$$

Therefore $$g = \frac{2M + m}{m} a.$$

Raise the mass P, Fig. 58, p. 103, and putting on the rider observe as in Experiment 17 the time taken to fall through some convenient distance, say 2 metres; let this distance be s cm. and the time of fall t seconds, then

$$s = \tfrac{1}{2}at^2.$$

Hence $$a = \frac{2s}{t^2},$$

$$g = \frac{2M + m}{m} a = \frac{2(2M + m)s}{mt^2} \text{ cm. per sec. per sec.}$$

The quantities on the right-hand can all be determined and hence a value can be found for g.

In an experiment the following values were obtained. It was found that the masses traversed 210 cm. in 10 seconds.

Hence $$a = \frac{2 \times 210}{10^2} = 4 \cdot 2 \text{ cm. per sec. per sec.}$$

The masses were $M = 1030$ grammes, $m = 9$ grammes.

Hence $$g = \frac{2 \times 1030 + 9}{9} \times 4 \cdot 2$$

$$= 966 \text{ cm. per sec. per sec.}$$

Sources of error.

The main sources of error are two. (1) The pulley has mass which though small may be appreciable; thus the whole rate of change of momentum is not $(2M + m)\,a$, but this quantity together with something depending on the pulley. Now the outer edge of the pulley moves with the same velocity as the descending masses, if we call this v we may represent the momentum of the pulley by $M'v$, where M' is not the mass of the pulley but is a mass which if concentrated in the rim of the pulley and moving with its actual velocity would have momentum equal to that of the pulley. M' is clearly less than the mass of the pulley, for some parts of the pulley are moving with a velocity less than v. Its exact value will depend on the distribution of mass in the various parts of the pulley. If, as is usual, nearly the whole mass is in the rim, M' will not be much less than the mass of the pulley. The rate at which the momentum of the pulley is changing is $M'a$. Hence the left-hand side of the equation of motion would be

$$(2M + m + M')\,a.$$

The numerator of the fraction giving the value of g should be increased by M' and the value of g should be larger.

(2) Again, owing to the friction the force acting is not mg but something less; the acceleration observed is less than it would be if there were no friction, the value of g found is in consequence too small. For a method of correcting for this, see Glazebrook and Shaw, *Practical Physics*, § 21.

EXAMPLES.

LAWS OF MOTION.

1. Calculate the momenta of a mass of 1 kilogramme when moving with the following velocities:

(i) 1 metre per second, (ii) 75 cm. per hour,
(iii) ·1 mile per minute, (iv) after falling 500 metres.

2. Compare the momenta of a bullet whose mass is $\frac{1}{2}$ an oz. moving with a speed of 1000 feet per second and of a mass of 60 kilogrammes whose speed is 1 kilometre per minute.

3. A mass of 60 kilogrammes acquires in moving for one minute a speed of 1 kilometre per minute. Find the acceleration and the impressed force, stating clearly the units in which each is measured.

4. Compare the impressed forces (supposed uniform) on the two bodies mentioned in question 2, assuming the bullet to have gained its speed in $\frac{1}{8}$ of a second and the large mass in 5 minutes.

5. Find the momentum of the Earth, taking its mass as 5×10^{27} grammes and assuming it to describe a circle of radius 92000000 miles in a year.

6. A bullet whose mass is 150 grammes, moving with a speed of 500 metres per second, strikes and remains imbedded in a lump of soft wood whose mass is 25 kilogrammes suspended by a string 1 metre long. Through what height does the wood swing?

7. A cricket ball moving with a speed of 30 feet per second is hit to square leg, and after the blow moves with double the speed. Find the impulse, and assuming contact with the bat to last for $\frac{1}{30}$ of a second, find the average force.

8. A spiral spring in expanding through 25 cm. can exert an average force equal to the weight of 1 kilogramme. Find the velocity it will produce in a mass of 10 grammes with which it is in contact through the whole distance.

9. A particle starting from rest is acted on by a force equal to the weight of ten pounds. After twelve seconds the velocity is five yards a second, find the mass and the weight of the particle.

10. A constant force acts on a particle in the direction of its motion, state the relation connecting the increase of momentum and the time during which the force has acted.

11. A mass of 10 lb. moving with a velocity of 25 feet per second is stopped by a uniform force in a distance of 50 feet: find the force.

12. A force equal to the weight of 10 grammes acts on a mass of 27 grammes for 1 second: if the value of g be 982, find the velocity of the mass and the space it has travelled over. At the end of the first second the force ceases to act; how much further will the body move in the next minute?

13. A force equal to the weight of 10 lb. acts upon a mass of 8 lb., the mass moves vertically upwards with uniform acceleration. What will be its velocity at the end of the third second starting from rest?

14. A force equal to the weight of 12 lb. acts on a heavy body which moves vertically upwards with uniform acceleration. If the body passes over 18 feet in three seconds starting from rest, find the mass of the body.

15. A force of a pound weight acting upon a certain body for a minute generates in it a velocity of 60 miles an hour: find the mass of the body.

16. How far will a railway carriage starting with a velocity of 60 miles per hour run on level rails if the resistance be ·002 of its weight?

17. A train going at 30 miles per hour pulls up in 200 yds.; what is the direction and magnitude of its acceleration?

If every wheel skids, find the resistance in terms of the weight, supposing the line to be level.

18. Describe some accurate method of determining the value of the acceleration due to gravity. How would you arrange an experiment to make a body fall with an acceleration each second of one foot per second?

19. What is the relation between the force producing motion in a given mass and the motion produced? How would you verify this relation?

20. A mass has its velocity changed (i) from rest to 10 feet per second, (ii) from 10 feet to 20 feet per second. Compare the magnitudes of the forces required, the time occupied in the change being in each case 5 seconds.

21. Describe an experiment to shew that the rate of change of momentum of a moving body is proportional to the force producing motion. What force is needed to reverse the motion of a mass of 1 lb. moving with a speed of 100 feet per second, supposing the whole change to take place in 1 second?

22. A weight of 100 lb. is placed on a smooth horizontal table; what force acting horizontally for three seconds will generate in it a velocity of 64 feet per second?

23. Shew that, if the force on a body be taken to be numerically expressed by the product of the numbers expressing its mass and acceleration respectively, the unit of force is dependant on the units of mass, length, and time. Indicate the unit of force thus defined when the gramme, centimetre, and second are the units of mass, length, and time respectively.

24. Describe Atwood's Machine. How would you prove by means of it that the space described from rest by a body acted on by a uniform force varies as the square of the time?

25. Two masses of 8 and 10 ounces are connected by a string passing over a pulley; find the tension of the string when they are in motion, and the space described in 4 seconds.

26. A spiral spring, which for every millimetre of extension requires a force of 20 grammes weight, is hung up by one end and a mass of 50 grammes is attached to the other end by a long string. If the mass is raised and allowed to fall so that it travels a distance of 30 centimetres before the string becomes tight, find what extension of the spring will be produced.

27. Two masses of 8 lb. and 10 lb. are supported by a fine string over a smooth pulley. After falling through 5 feet the 10 lb. weight strikes the floor, find the impulse on the floor.

28. Masses of 1 and 3 lb. hang from the two ends of a fine string suspended over a smooth pulley. At what rate will they be moving at the end of 1 sec. after they are set free?

29. Weights of five and ten grammes are connected by a string which passes over a pulley: if the weights are allowed to fall, find their velocity when the heavier weight has descended through a metre.

30. Find the tension of a rope which draws a carriage weighing 1 ton up an incline of 30° with a velocity which increases by 1 foot per second per second.

31. When one weight lifts another by means of a string passing over a fixed pulley without friction, find the tension of the string and the velocity produced in one second.

If the first weight be 4 lb. and the second 2 lb. what is the tension of the string in lb.-wt.? and what the velocity produced per second in feet per second?

32. Masses of 3 lb. and 3 lb. 1 oz. respectively are attached to the ends of the string of an Atwood's machine and move from rest during 4 seconds; the 1 oz. is then removed and it is found that the masses move over 8 feet in the next 5 seconds. Find the numerical value of gravity which results from these experiments.

33. Each of the masses attached to the ends of the string of an Atwood's machine are 150 grammes; if the inertia of the pulley be taken as equivalent to an additional mass of 25 grammes, find the acceleration when a mass of 5 grammes is added to one side. The value of g may be taken as 980 cm. per sec. per sec.

34. Two weights each of 5 lb. are tied to the two ends of a long cord which hangs over a pulley in a vertical plane. An additional weight of 2 lb. is suddenly placed on one end. Find the acceleration of the weights, and also the velocity and amount of displacement after 3 seconds, neglecting friction and inertia of the pulley and the stiffness of the cord.

35. How may Atwood's machine be used to find the acceleration due to gravity? The masses on either side are 250 grammes; if the inertia of the pulley be taken as equivalent to an additional mass of 50 grammes, find the acceleration when a mass of 5 grammes is added to one side. The value of g may be taken as 980 cm. per sec. per sec.

36. A heavy particle slides down a smooth inclined plane, starting from rest at the top, the height of the plane being h and the length l feet. Find the acceleration of the particle, and the time it will take to reach the bottom.

With what velocity must a particle be projected down a plane 12 feet in height and inclined to the horizon at an angle of 30°, so as to reach the bottom in one second?

37. A heavy particle is projected up a smooth inclined plane whose height is h, and length l, with such a velocity that it just reaches the top of the plane. Find the acceleration of the particle and the time that it takes to mount.

With what velocity must a particle be projected up a plane 10 feet in height and inclined to the horizon at an angle of 30°, so as to reach the top in one second?

38. The line AB is vertical, and ACB is a right angle. Shew that the time of sliding down either AC or CB, supposed smooth, is equal to the time of falling down AB.

39. Let PQ be a chord of a vertical circle whose highest point is A and centre O. Then if the time of descent down this chord be half of that down the vertical diameter, shew that

$$\tan \tfrac{1}{2} AOP : \tan \tfrac{1}{2} AOQ :: 3 : 5.$$

40. Distinguish between a poundal and the weight of one pound.

What is the experimental evidence for the statement that the weight of a body is proportional to its mass?

41. Distinguish between mass and weight.

If the weight of a certain mass be represented by 15 at a place where a body falls through 64 feet in 2 seconds, what will be the weight of the same mass at a place where a body falls through 176 feet in 3 seconds?

42. The time of falling down a smooth inclined plane is twice that down the vertical height of the plane. Find the ratio of the length of the plane to its height.

43. If the wheel in an Atwood's machine is so stiff that a weight P on one side will just support nP on the other, find the acceleration when the weights are P and $n'P$ $(n' > n)$.

44. Find the change of momentum of a body of mass m which is initially moving with velocity u, and then has its velocity deflected through an angle α, but without change of magnitude.

Two particles A and B of the same mass m are connected by a light string and rest with the string just tight on a horizontal table whose coefficient of friction is μ. A uniform force P (greater than $2\mu mg$) begins to act on the particle A in the direction BA. Find the acceleration of either particle and the tension of the string at any time.

45. A mass of 10 grammes resting on a smooth table is connected by a string passing over a pulley with an equal mass which hangs vertically. The first mass is 1 metre from the edge; find the velocity of the system at the moment at which it is dragged over.

46. A mass of 200 grammes can just be dragged up an inclined plane by a mass of 100 grammes which is connected to it by a string passing over a pulley and hangs vertically from the top of the plane. Find the inclination of the plane; find also the velocity produced after the system has moved through 1 metre, if a mass of 5 grammes be placed on the smaller mass.

47. A mass of 1 lb. slides down an inclined plane whose height is half its length and draws a mass of 5 lb. along a smooth horizontal table level with the top of the plane. Find the acceleration; find also the momentum of the whole after the masses have moved 1 yard.

48. Two masses of 5 lb. and 10 lb. respectively are placed on two inclined planes of the same height and the angle 30°; the masses are connected by a fine string passing over a smooth pulley at the top of the two planes. Find the acceleration, the tension of the string, and the distance traversed in 5 seconds.

49. An engine draws a train whose mass is 75 tons up a slope of 1 in 50. If the resistance of the rails be $\frac{1}{80}$ of the weight, find the force exerted by the engine in order that the speed may be constant. The speed at the bottom of the slope is 25 miles an hour: if the engine stops working how far will the train run?

50. A mass of 10 kilogrammes slides down a rough plane which rises 1 in 10: find the resistance if the speed remains constant.

51. What is meant by the equation $W = Mg$?

A lift is rising with an acceleration of 8 feet per second per second; what pressure would a man scaling 16 stone exert on the floor of the lift?

52. A particle, starting from rest, slides down a smooth plane inclined at an angle of 45° to the horizon in 6 seconds; find the length of the inclined plane.

53. Two equal masses are at rest side by side. One moves from rest under a constant force F while at the same instant the other receives an impulse I. Shew that they will again be side by side after a time $\frac{2I}{F}$.

54. A particle whose mass is 10 lb. is moving with a velocity of 30 feet per second. If it is brought to rest in 100 feet by applying a constant resistance, find the magnitude of the resistance.

55. A particle whose mass is 12 lb. is moving with a velocity of 25 feet per second. If a constant resistance equal to a weight of 15 oz. is applied to stop it, find how far it will travel before it comes to rest.

56. A weight of 7 lb. is placed on a smooth horizontal board and connected by strings passing over smooth pulleys at each end of the board with weights of 5 lb. and 4 lb. respectively. Find the acceleration of the weights and the tensions of the strings.

57. If two masses m, m' are connected by a string, whose mass can be neglected, passing over a smooth fixed pulley, find the tension of the string.

Two bodies, of mass 2 lb. and 30 lb. respectively, lie on a smooth horizontal table whose height above the floor is 27 inches. The bodies are connected by an inextensible string, whose length is not less than 27 inches, and, when the string is taut, the smaller mass is dropped through a hole in the table. Find when it reaches the ground.

CHAPTER VIII.

THE THIRD LAW OF MOTION. ENERGY.

99. Action and Reaction. In the Third Law of Motion Newton stated that *To every action there is always an equal and opposite reaction, or the mutual actions of two bodies are always equal and opposite.*

The Law is based on observation and experiment, in considering it however we are at once met by the question What is meant by **Action**? We can learn from Newton's own illustrations what he understood by the term.

"If a man presses a stone with his finger," he says, "his finger also is pressed by the stone. If a horse draws a stone by means of a rope the horse is drawn equally towards the stone, for the rope stretched between the two will urge the horse towards the stone and the stone towards the horse; and this will impede the progress of the one as much as it helps that of the other."

"If a body impinging on a second body changes the motion of that body in any manner by the force it exerts, it will itself undergo the same change in its own motion in the opposite direction through the force exerted by the second body (because of the equality of the mutual pressure)."

"It is the amount of *momentum*, not of *velocity*, interchanged in these actions which is equal, at least in bodies which are otherwise unimpeded. For the changes of velocity which take place also in opposite directions are reciprocally proportional to the [masses of the] bodies, since the change of

momentum is the same for the two. The law is also true for the case of attractive forces."

100. Illustrations of the Third Law. These illustrations of Newton's make it clear that the action and reaction contemplated by him was the *Interchange* of *Momentum* between two bodies. We shall see later that, in a scholium attached to the law, he interprets the terms in another sense in which they are equally true.

If then we are to mean by **Action** *Gain of Momentum*, the experiments on impact described in Sections 51—61 afford a verification of the law.

Moreover, when two bodies are unimpeded in their action on each other, not only is the whole gain of momentum of the one equal to the whole loss of the other, but, during the time of transfer, the rate at which the one gains momentum is equal to the rate at which the other loses it. The impressed force on the one is equal and opposite to that on the other. The Moon has an acceleration towards the Earth and the Earth towards the Moon, astronomical observations shew that these accelerations are inversely proportional to the masses of the Moon and the Earth respectively. For Moon and Earth the equation $ma = m'a'$ holds, m and m' being the respective masses, a and a' the respective accelerations. Each gains momentum at the same rate. An experiment, also due to Newton, may illustrate this point. Float a magnet on a block of wood in a large[1] vessel of water; float a piece of soft iron on a second block of wood. Place the iron at a little distance from the magnet and in such a position that the magnet may point directly towards it. Hold the two at rest until the water is quite still, then release them simultaneously. The magnet will draw the iron to itself, the two blocks of wood will impinge and then come to rest[2] on the water.

The two blocks gain momentum in opposite directions when free to move, this gain however is the same for each, the force

[1] Large because otherwise capillary action at the sides of the vessel may affect the result.

[2] It is desirable in performing this experiment to arrange that the impact may be direct, otherwise the blocks of wood may be made to spin round in the water.

on the one is equal and opposite to that on the other; after impact the system remains without momentum.

The following also affords us an illustration of this law. When a gun is fired it kicks, unless held close to the shoulder it bruises it. If the gun be held firm, it and the man constitute, so far as the action is concerned, but one body, the mass of which is much greater than that of the bullet; the velocity acquired by this body will be small compared with that of the bullet, but the momentum of the bullet and of the gun and man combined will be equal and opposite. We can shew by the following experiment that this is so.

The gun is attached with its barrel in a horizontal position to a massive block of wood which can swing about a horizontal axis above the gun and at right angles to the barrel; the whole constitutes a pendulum; as with the simple pendulum if the gun be displaced from its equilibrium position its velocity can be calculated by observing the arc through which it swings, the velocity of the bullet as it leaves the barrel can also be found. Suppose the gun loaded and fired when at rest at the lowest point of its swing; observe the velocities of the gun and pendulum and of the bullet and calculate the momentum of each: it will be found that they are equal.

A simpler experiment of the same character is as follows.

EXPERIMENT 22. *To illustrate by experiment the third law of motion.*

Suspend two balls, Fig. 64, of different masses, so that their centres may be at the same level and equally distant from their points of support; compress a fairly strong spiral spring and tie it with thread so that the spires are as close together as possible. Place it between the balls; release the spring by cutting or burning the thread, the balls will move in opposite directions; determine their velocities by observing the distances through which each ball swings, it will be found that the velocities are inversely proportional to the masses of the balls. Each ball acquires the same amount of momentum.

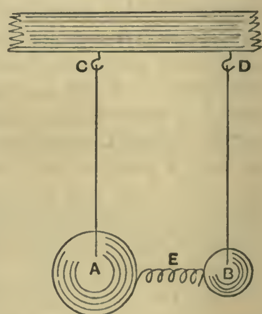


Fig. 64.

Hicks' ballistic balance already described may be used for this experiment, with this apparatus the velocities are easily measured.

One case mentioned by Newton needs a little further consideration; when a horse drags a load or tows a boat how is it that the horse is pulled back as much as the load is pulled forward? The horse acquires momentum by the action between his feet and the road. If he is moving with uniform speed and we neglect the relative motion of the various parts of his body the whole of the momentum so acquired is transferred by the rope to the boat. The passage of the boat through the water is resisted and the boat loses momentum to the water, this loss is equal to the gain it acquired from the horse; thus the momentum gained by the horse appears in the water wave which follows the boat. Now the Earth has lost momentum—estimated in the direction in which the horse is moving —equal to that gained by the horse: by the actions just described the water forming part of the Earth has gained the same amount in the direction of the horse's motion, thus on the whole there is neither loss nor gain. If the boat is gaining speed the horse acquires from the ground rather more momentum than is transmitted by the rope to the boat.

***101. Attraction and the Third Law.** This equality of action and reaction holds also in the case of the **Attractions** between the various portions of the same body.

For consider a body like the Earth, imagine it free from external action, and suppose it divided into two parts A, B, Fig. 65, by a plane CD; if it be possible let A gain momentum from B faster than B gains it from A, then the whole system is continually acquiring momentum in the direction from B to A and will move off with continually increasing velocity into space in the direction BA; this is contrary to the first law for the body is free from external action.

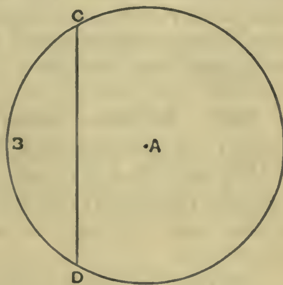


Fig. 65.

Example. As an example of the third law consider the case of a shell exploding in the air. The fragments are projected in various directions but the total gain of momentum of the various parts is zero; the momentum gained in one direction by some parts is equal to that gained in the opposite direction by others. Thus if the shell burst into two equal pieces, if its original velocity be 100 feet per second, and the velocity after fracture of one part in the same direction be 150 feet per second we have, calling m the mass of the whole shell and v the velocity of the other part,

$$m \times 100 = \frac{m}{2} \times 150 + \frac{m}{2} v.$$

Hence $v=50$, the one half gains $25m$ units of momentum, the other loses the same amount and continues to move forward with a velocity of $100-50$ or 50 feet per second.

We see by these and similar examples how to interpret the third law of motion if we give to action the meaning of change or rate of change of momentum.

102. Conservation of Momentum. In all these cases then **Action is Transference of Momentum.** Whenever a transference of momentum takes place between two bodies the loss of the one body is equal to the gain of the other.

From this point of view the third law expresses the **Conservation of Momentum.**

We can observe the facts that the one body gains momentum while the other loses it; we call the rate at which this transference takes place **Force**, *and when the transference is going on we say that* **Force acts** *between the two bodies, its action on the two being equal and opposite.*

103. Action.—Energy. But the third law as Newton pointed out contains more than this.

We shall find that, when bodies act on each other, another quantity (**Energy**) besides Momentum is transferred, and that in this case too there is neither gain nor loss on the whole transaction.

In the Scholium to the laws of motion, Newton calls attention to the importance of the quantity obtained by multiplying force and the displacement of its point of application. He writes after giving some examples—"By these I wished to shew how wide-spread and how certain is the third law of motion. For if the action of an agent be measured by the product of its force and displacement[1], and similarly the reaction of the resisting body be measured by the sum of the products of the resistances and their several displacements, then whether the resistances arise from friction, cohesion, weight or acceleration, in all cases action and reaction

[1] The word actually employed is velocity, but the velocities concerned are those which occur during the same moment, they are measured therefore by the displacements.

in every kind of machine will be equal." **Action** then in the third law may be measured by the *product of a force and the displacement of the point at which it is applied*; this displacement however as Newton points out is to be estimated in the direction of the force.

In other words, when momentum is being transferred to a body at a given point, the product of the rate at which the body is gaining momentum and the displacement of the point at which the transference is taking place may measure the Action which is considered in some aspects of the third law.

The meaning and importance of this statement will however be better appreciated after some preliminary explanations and definitions.

104. Work and Energy. When momentum is being transferred to a body and the point at which the transference is taking place is in motion, we say in general that **Work** is being done.

Now let F be the amount of momentum transferred per second—the force; let A, Fig. 66, be the point at which the transference is taking place—the point of action of the force. Let AB be the direction in which the momentum is being transferred—the direction of the force. Let A be displaced to A' and draw $A'A_1$ perpendicular to AB to meet it in A_1.

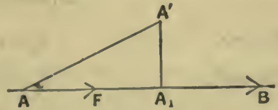


Fig. 66.

Then the work done is measured by the product $F \times AA_1$.

The actual displacement of the point A is AA', this can be resolved into AA_1 in the direction of the force and A_1A' at right angles to the direction of the force, when this is done AA_1, the component of the displacement in the direction of the force, is spoken of as the displacement in the direction of the force and the work done is the product of the force into the displacement in the direction of the force.

105. Measurement of Work.

DEFINITION. *When* Momentum *is being transferred to a body at a point which is in motion the* Work *done is measured by the product of the rate at which the body is gaining momentum at that point multiplied by the component of the displacement in the direction of the momentum.*

Or, in other words. The product of a Force into the component in its own direction of the displacement of its point of application measures the Work done.

Thus let F be the force, let the point of application of the force be displaced a distance s in the direction of the force; the work done U is given by the equation

$$U = Fs.$$

When the actual displacement s is not in the direction of the force we resolve it, as in Figure 66 above, into s_1 in that direction and s_2 at right angles to the direction, in this case the work done is given by Fs_1. If the angle A_1AA' be called θ, then since $A'A_1A$ is a right angle we have from Figure 66,

$$A_1A = AA' \cos\theta.$$

Hence $$s_1 = s\cos\theta.$$

Therefore $$U = Fs\cos\theta.$$

Now in this expression $F\cos\theta$ is the component of the force in the direction of displacement. Thus we see that the work is found by multiplying together either the force and the component of the displacement in the direction of the force or the displacement and the component of the force in the direction of the displacement.

Hence

Work = Force × component of displacement in the direction of the force.

= Displacement × component of the force in the direction of the displacement

$$= Fs\cos\theta.$$

If, as in Fig. 67, the displacement AA' be at right angles to the force then the displacement in the direction of the force is zero, thus the work done is zero.

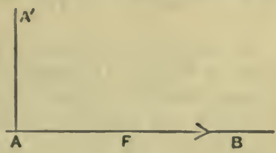


Fig. 67.

Hence, when the motion of the point of application is at right angles to the direction of the force, no work is done.

Again, the component of the displacement with which we are concerned may be as AA_1 in Fig. 68 in the same direction as that in which the force acts or it may be as AA_1 in Fig. 69 in the opposite direction to that of the force. In the first case, Fig. 68, work is done *on* the body *by* the force, in the second, Fig. 69, work is done *by* the body *against* the force.

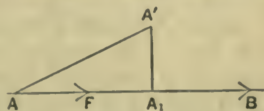


Fig. 68.

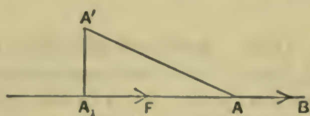


Fig. 69.

Thus when a body is gaining momentum at any point and when the direction of motion of the point is the same as that of the gain of momentum work is done *on* the body; when the direction of motion of the point is opposite to that of the gain of momentum work is done *by* the body; when the direction of motion is at right angles to that of the momentum no work is done.

Hence, if a body, acted on by gravity only, move in a horizontal direction with uniform speed no work is done, if the body be raised work is done on the body against its weight by the agent raising it, if the body be allowed to fall from a height work is done by its weight.

When a body of mass m is raised a height h the upward impressed force is mg and the work done is mgh.

In making this statement it is supposed that the body is raised very slowly. The upward force is really greater than mg, otherwise the body would not move, but a force which is just in excess of mg will raise it; if the excess be extremely small the motion will be exceedingly slow, and

the work done will differ from mgh by a very small quantity which we may neglect. If the body is drawn up with a run more work will be necessary. See Section 112.

It should be noticed that two factors are necessary to measure work. It depends on the product of the force or rate of transference of momentum and of the displacement. If we know the amount of work done, we cannot calculate the force unless we also know the displacement: a small force working through a large distance may do as much work as a large force working through a small distance, the work is the product of the two.

Again, the work done does *not* depend on the time in which it is done. An engine which can raise a ton a foot in the course of a year will then have done as much work as one which raises a ton the same distance in a second; the amount of work done depends solely on the product of the force and the displacement, and is quite independent of the time taken to do it.

106. Unit of Work. Consider now a body which is gaining F units of momentum per second; if the body be displaced a distance s in the direction of this momentum, then the work done is U, where

$$U = F \, . \, s.$$

If then the force or rate of gain of momentum be unity and the displacement also be unity the work done is unity.

Thus the **Unit of Work** is done when a particle on which unit force is acting is displaced unit distance in the direction of the force.

The value then of the unit of work depends on the unit of force and on the unit of length.

107. The C. G. S. Unit Work. On the C. G. S. system the unit force is a Dyne and the unit distance a Centimetre; the C. G. S. unit of work then is done when a particle on which 1 Dyne is acting is displaced 1 Centimetre in the direction of the force, this unit of work is called an **Erg.**

DEFINITION. *One* Erg *is the work done when the point of application of a force of* 1 *dyne is moved* 1 *centimetre in the direction of the force.*

The Erg is a very small unit of work for a dyne is a very small unit of force, being rather less than the weight of 1 cubic millimetre of water. Hence, the erg is rather less than the work done in raising a cubic millimetre of water 1 centimetre. For this reason ten million ergs are taken as the practical C. G. S. unit of work and are called a **Joule.**

Thus we have

$$1 \text{ Joule} = 10,000,000 = 10^7 \text{ Ergs}.$$

108. The F. P. S. Unit Work. On the F. P. S. system the unit force is one poundal. Unit work then is done on this system when the point of application of a force of 1 poundal is moved through 1 foot. This unit is called the Foot-poundal.

DEFINITION. *One* Foot-poundal *is the work done when the point of application of a force of* 1 *poundal is moved* 1 *foot in the direction of the force.*

109. Gravitational units of work. Another unit of force used is the weight of a body whose mass is 1 gramme. When the point of application of such a force is moved 1 centimetre, 1 centimetre-gramme unit of work is done. Since the weight of a gramme contains g dynes we see that 1 centimetre-gramme unit of work contains g ergs.

Moreover since g depends on locality the centimetre-gramme unit of work is different at different points of the Earth, the erg is the same everywhere.

DEFINITION. *One* centimetre-gramme unit of work *is done when the point of application of a force equal to the weight of* 1 *gramme is moved* 1 *centimetre in the direction of the force.*

Hence, 1 centimetre-gramme unit of work

$$= g \text{ ergs} = 981 \text{ ergs},$$

if we take g as 981 centimetres per second per second.

Again, the weight of a mass of 1 pound is taken sometimes as the unit of force. When this is done the corresponding unit of work is the foot-pound.

DEFINITION. *One* **foot-pound unit** *is the work done when the point of application of a force equal to the weight of* 1 *pound is moved* 1 *foot in the direction of the force.*

Since the weight of 1 pound contains g poundals we see that one foot-pound is equal to g foot-poundals, and since in feet per second per second g is 32·2, we see that

$$\begin{aligned} 1 \text{ foot-pound} &= g \text{ foot-poundals} \\ &= 32{\cdot}2 \text{ foot-poundals.} \end{aligned}$$

A foot-pound of work is done when a mass of 1 pound is raised 1 foot.

The work done in raising a kilogramme through one metre is 1000×100 or 100,000 centimetre-gramme units. Since each of these contains 981 ergs the work is 98,100,000 ergs.

110. Rate of Working. The rate at which work is done is called **Power.**

DEFINITION. **Power** *is measured, when uniform, by the work done per second, when variable by the ratio of the work done in an interval of time to that interval when it is sufficiently small*[1].

Thus the powers of the two engines mentioned in Section 105 are very different. The second has a power of 1 foot-ton or 2240 foot-pounds per second, while the first, since there are 31536000 seconds in the year, has a power of 2240/31536000 foot-pounds per second.

A **Horse-Power** is the name given to a unit of power in common use.

DEFINITION. *When* 550 *foot-pounds of work are being done per second the rate of working is* 1 **Horse-Power.**

[1] See Section 22.

Thus the second engine is one of rather over 4 horse-power (really 224/55 horse-power).

The rate of working in common use on the C.G.S. system is **The Watt.**

DEFINITION. *When* 1 *Joule* (10^7 *ergs of work*) *is being done per second the* **Rate of Working** is 1 **Watt.**

Thus 1 Watt is 10^7 ergs done per second. We can shew[1] that a Joule is about ·737 of a foot-pound, so that a Watt is ·737 foot-pound per second. *Thus a Horse-power is* 746 *Watts.*

111. Measurement of Power. Since work is measured by force multiplied by displacement, if the force be constant the rate of working is measured by force multiplied by rate of displacement. Now rate of displacement is velocity.

Hence **Power** is measured by *the product of a force into the velocity of its point of application measured in the direction of the force.* In other words, the rate at which work is being done on a particle is the product of its rate of gain of momentum and the component of its velocity measured in the direction in which it is gaining momentum. Thus if F be the force impressed on a particle and v_1 the component of its velocity in the direction of F. Then Rate of Working $= Fv_1 = Fv\cos\theta$, if v be the velocity and θ the angle between the directions of v and F.

112. Expressions for Work and Power. The expressions which have been found for Work and Power may be put into various forms. Thus

PROPOSITION 27. *To shew that if a body of mass* m *acquire a velocity* v *in moving with constant acceleration in a straight line from rest through a space* s *the work done is* $\frac{1}{2}$mv².

Let F be the impressed force, a the acceleration, U the work. Then we have $F = ma$ and $v^2 = 2as$.

Hence $U = Fs = mas = \frac{1}{2}mv^2$.

Again, if the initial velocity be u and not zero, we have $v^2 - u^2 = 2as$.

Hence $U = Fs = mas = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$.

[1] See Section 112, Example 4.

The units in which this result is given will depend on those in which m and s are measured, for instance, on the C.G.S. system m is in grammes and s in centimetres; since we apply the equation $F=ma$ the force is in dynes, hence the work Fs is in centimetre-dynes, or ergs.

Thus the work required to give to a mass of m grammes a velocity of v centimetres per second is $\frac{1}{2}mv^2$ *ergs*. If the mass be in pounds, the space in feet, then the work is in foot-poundals. If we wish to use gravitation measure we must remember that a dyne is $1/g$ of the weight of one gramme.

Hence $\frac{1}{2}mv^2$ ergs is $\frac{1}{2}mv^2/g$ centimetre-grammes of work and $\frac{1}{2}mv^2$ foot-poundals is $\frac{1}{2}mv^2/g$ foot-pounds where g is 981 cm. per sec. per sec. in C.G.S. units or 32·2 feet per sec. per sec. in F.P.S. units.

PROPOSITION 28. *To find expressions for the rate at which work is being done on a particle of mass* m, *moving in a straight line with constant acceleration* a.

Let v be the velocity of the particle at any moment, F the force, t the time from rest, and s the space traversed.

Then we have

$$\text{Power} = \text{Rate of Working} = Fv = mav$$

$$= \frac{mv^2}{t} = ma^2t = \frac{2mas}{t} = ma\sqrt{2as}.$$

Examples. (**1**). *Find in the various units the work done on a mass of* 1 *cwt. when lifted through* 100 *feet.*

Since 1 cwt. = 112 pounds,

work = 112 × 100 foot-pounds,

= 112 × 100 × 32·2 foot-poundals.

Also 1 lb. = 453·6 grammes,

1 foot = 30·48 cms.

Thus work = 112 × 453·6 × 100 × 30·48 centimetre-grammes

= 112 × 453·6 × 100 × 30·48 × 981 ergs,

and this reduces to about $1·519 \times 10^{11}$ ergs or 15190 Joules.

(**2**). *This same mass is allowed to fall from a height of* 100 *feet. Calculate the work done by gravity* (*a*) *after it has fallen* 50 *feet*, (*b*) *when it has reached the ground, and determine in each case the rate at which work is being done.*

The rate at which the mass is gaining momentum or the force is 112 × 32·2 poundals.

Thus the work which has been done in 50 feet is 112 × 32·2 × 50 foot-poundals, and in 100 feet it is twice as much, or 112 × 32·2 × 100 foot-

poundals. This last expression is the same as the work done in lifting the body to the height of 100 feet.

The rate of working is the product of the velocity and the rate at which momentum is being communicated. After falling 50 feet the velocity is $\sqrt{2 \times 32{\cdot}2 \times 50}$ feet per second, and the force is $112 \times 32{\cdot}2$ poundals, thus the rate of working is

$$112 \times 32{\cdot}2 \times \sqrt{2 \times 32{\cdot}2 \times 50} \text{ foot-poundals per second.}$$

This reduces to $2{\cdot}033 \times 10^5$ foot-poundals per second; if we wish to work in foot-pounds per second we have for the power $112 \times \sqrt{2 \times 32{\cdot}2 \times 50}$ or

6349 foot-pounds per second.

Dividing this by 550 we find for the horse-power 11·54.

Thus when a body of 1 cwt. in mass has fallen freely through 50 feet, work is being done on it at the rate of 11·54 horse-power.

When the body has fallen 100 feet we shall have to substitute 100 for 50 in the above formulæ,—the velocity will be $\sqrt{2 \times 32{\cdot}2 \times 100}$ feet per second, and we find for the rate of work 8964 foot-pounds per second or 16·17 horse-power.

(**3**). *Two bodies* 1·5 *kilos and* 1 *kilo in mass respectively, suspended by a fine string over a pulley are free to move. Find the work done* 5 *seconds after motion has commenced, and the rate at which it is then being done.*

The acceleration is given by

$$a = \frac{1{\cdot}5 - 1}{1{\cdot}5 + 1} g.$$

Thus $a = g/5$ cm. per sec. per sec.

The rate at which the system gains momentum in the downward direction is $(1500 - 1000)\, g$ dynes, and this reduces to $500g$ dynes.

In 5 seconds the velocity (at) is $5 \times g/5$, or g cm. per second, and the space traversed $\frac{1}{2}at^2$ or $25g/10$ cm.

Thus the work done is

$$\frac{500g \times 25g}{10} \text{ or } 50 \times 25 \times g^2 \text{ ergs.}$$

This reduces to $120{\cdot}3 \times 10^7$ ergs or 120·3 Joules.

The rate of working being the product of the force and the velocity is

$$500g \times g \text{ ergs per second,}$$

or

$$48{\cdot}12 \times 10^7 \text{ ergs per second.}$$

This is 48·12 watts.

(4). *Find the value of an erg in foot-poundals and of a horse-power in watts.*

$$1 \text{ erg} = \frac{1}{981} \text{ centimetre-gramme unit,}$$

$$1 \text{ centimetre} = {\cdot}03281 \text{ feet,}$$

$$1 \text{ gramme} = {\cdot}002205 \text{ lbs.}$$

$$1 \text{ erg} = \frac{{\cdot}03281 \times {\cdot}002205}{981} \text{ foot-pounds.}$$

This reduces to ·0000007374 foot-pound.

Hence 1 Joule is ·7374 foot-pound.

Thus $$1 \text{ foot-pound} = \frac{1}{{\cdot}7374} \text{ Joules}$$
$$= 1{\cdot}356 \text{ Joules.}$$

$$1 \text{ horse-power} = 550 \times 1{\cdot}356 \text{ Joules per second}$$
$$= 745{\cdot}8 \text{ watts}$$
$$= \text{approximately } \tfrac{3}{4} \text{ of a kilowatt.}$$

113. Measurement of Work. In the above examples we have shewn how to calculate the work done in various cases in which a particle—or a body which we may treat as a particle—is displaced in the line of action of the force; we will now consider some cases in which the displacement is oblique to the force. Suppose then that a body of mass m is moved from a point A to a point B, Fig. 70, where B is not vertically above A. Let the motion take place along the line AB and let us calculate the work done against gravity. The weight of the body is mg and its direction is vertical. If then we draw AC vertical and BC horizontal meeting in C, the displacement may be resolved into AC vertical in a direction *opposite* to that of the weight and BC horizontal in a direction at right angles to that of the weight. By the definition of work then the work done *against* the weight is $mg \times AC$ or mgh, if h is the vertical distance between A and B.

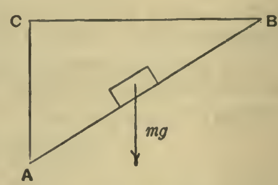


Fig. 70.

For the purpose of calculating the work, we may look upon the displacement as a vertical one $AC\,(=h)$ in which mgh units of work are done, and a horizontal one CB in which no work is done.

In raising a body from one point to another the work done against its weight depends only on the difference of level between the two points.

Moreover this result is independent of the path described by the body: if it be first moved vertically downwards its weight will do work but this will be cancelled by the work done against the weight in raising the body to its original position. If the path from A to B be a broken one as shewn in Fig. 71

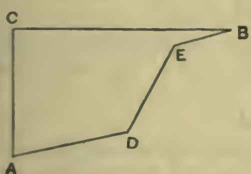


Fig. 71.

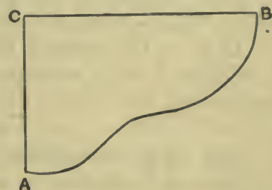


Fig. 72.

or a curved one as in Fig. 72, the work done is still the same, the vertical component of the whole displacement is in all cases AC or h and the work therefore is mgh.

Thus when the weight of the body is the only force considered, the work done in moving the body from one position to another against its weight depends only on the relative position of the two points—being proportional to their vertical distance apart—and not at all on the path of the particle between the points.

Now it can be shewn that a similar statement is true for a very large number of actually observed cases of motion. The work done on a body, which is gaining momentum by many of the processes which occur in nature, can be shewn to depend only on the initial and final positions of the body relative to such of its surroundings as influence its motion, and not at all on the path by which it has moved from one position to the other, or on the speed with which it has traversed this path.

There are some cases of motion for which this statement is not true. It is the fact however of its truth in many cases which gives to Work its great importance in Mechanics.

Action *then as used in the third law may mean the* Work done on a body, *reaction will be the work which a body can do in consequence of this, and the law states that these two, when all the forms of reaction are taken into consideration, are equal.*

In the *Statics* we shall have numerous examples of this principle and shall describe experiments arranged for its verification.

114. Motion of a body down a plane. Work. We have already found the value of the work done on a body by its weight when it is allowed to fall freely.

If m be the mass of the body, h the height through which it falls, and v the velocity it acquires, then U the work done is given by

$$U = mgh = \tfrac{1}{2}mv^2.$$

Let us now consider the work done by its weight on a body which is allowed to slide down a smooth inclined plane of height h. We know that the work done against its weight when the body is lifted from the bottom to the top of the plane is mgh.

PROPOSITION 29. *To find the work done on a body of mass* m *in sliding down a smooth inclined plane of height* h.

Let BA, Fig. 73, be the plane making an angle α with the horizon. Draw BC vertical and AC horizontal, let R be the force between the plane and the body; the weight of the body is mg.

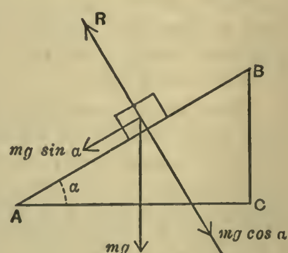


Fig. 73.

Then since the plane is smooth the direction of R is at right angles to it, the displacement of the body is along BA, at right angles to R, hence no work is done by the force R: the only force which does work is the weight mg, the displacement of

the body in its direction is h; hence the work done is mgh; thus an amount of work mgh is done on a body which slides down a smooth plane of height h.

Now let v be the velocity with which the body reaches the bottom, the acceleration parallel to the plane is $g \sin \alpha$: if l be the length of the plane we have a velocity v acquired in moving a distance l with acceleration $g \sin \alpha$.

Hence $\quad v^2 = 2gl \sin \alpha = 2gh, \quad$ for $h = l \sin \alpha$.

Thus $$mgh = \tfrac{1}{2}mv^2.$$

Hence the work done in sliding down the smooth plane which is equal to mgh is also given by $\frac{1}{2}mv^2$.

Moreover if the body be projected up the plane with velocity v it will just reach the top, and the work done against gravity will be mgh.

115. Work due to Gravity. We have just shewn that if a body be allowed to slide down a smooth inclined plane the velocity with which it reaches the bottom depends only on the height of the plane, and also that if the body starts up a second inclined plane with this same velocity it will rise to exactly the same height as that from which it fell.

Work is done by gravity on the body in sliding down; the body will rise again until this same amount of work is done against gravity.

Galileo discovered this relation between the velocity and height of fall and verified it by experiments which we will describe shortly.

We have deduced the above results on the supposition that the body slides down a *smooth flat* surface, a plane, so that its path is a straight line; it can be shewn that they are true if the surface be not flat but curved so that the path is a curve, not a straight line; for we may consider the curve as made up of a large number of very short straight lines inclined to each other at very small angles, the proposition is true for each of these lines; it is also true in the limit when they become a curve, though the proof of this would require some further consideration.

***116. Motion down a curve.** It is difficult to make observations on a particle *sliding* on a smooth curve;

no curve is perfectly smooth, and the corrections introduced by the friction are considerable.

We can however easily investigate motion in which the conditions are the same as on a curve. Thus, if a heavy body be suspended by a string and allowed to swing in a vertical plane it will move in a circle, the conditions of motion will be exactly the same as though it were sliding down the circular arc; the tension of the string acting at right angles to the path takes the place of the resistance of the curve. No work is done by this tension: if h be the vertical distance through which the body rises the work done is always mgh.

Consider now a body moving in such a circle; if by any means we can fix a point in the string, the body will continue to move in a circle but the radius of the circle will be less than before.

We can attain this result by allowing the motion to take place in front of a vertical wall or board; fix a nail or peg into the wall in such a way that the string may strike the nail, which will thus become the centre of the circle in

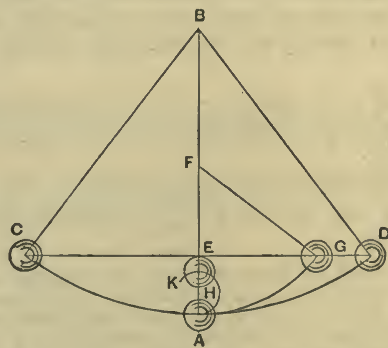


Fig. 74.

which the body will commence to move. Such an arrangement is shewn in Fig. 74; or again, if as in Fig. 75, we allow the string to unwrap itself off or to wrap itself on a curved surface such as FK, the path of the body will not be a

circle but will depend on the form of this curve. By properly adjusting this we can make the body to describe any path

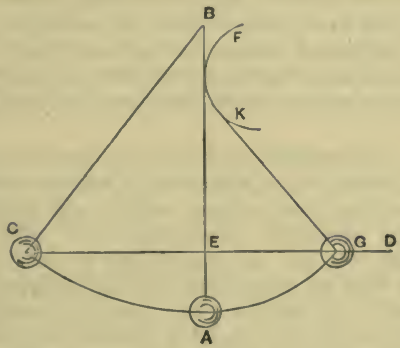


Fig. 75.

which we like and can thus investigate the motion of a body sliding down any smooth curve. In this way Galileo shewed that the *Height to which a body when moving under gravity on a smooth curve will rise depends only on the vertical height of its starting point above the lowest point of its path.*

EXPERIMENT 23. *To shew that a body moving under gravity in an arc of a vertical circle will ascend to the same height above the lowest point of the circle as that from which it started.*

Suspend a heavy body—an iron or lead sphere, some 6 or 8 cm. in diameter—by a long flexible cord such as a piece of waterproofed fishing-line about 2 metres long. Allow it to swing in front of a vertical wall or drawing-board about a point B, Fig. 74, and note the position C from which it starts. Observe the position D to which it rises at the end of its first swing and let A be the position it would occupy at rest. Join CD cutting BA in E, then it will be found that CD is horizontal[1]; the ball thus rises to the same height above A as the point from which it started.

[1] This statement is not quite strictly true, D will be a very little lower than C; the difference in height being due to the friction of the air for

Now drive a nail or peg into the wall as shewn at F a point in the vertical line AB, and again start the ball from C: when the string becomes vertical the portion BF is brought to rest, the ball proceeds to move in a circle AG with F as centre and rises to the position G before its motion stops. Observation will shew that G is in the horizontal line CD, the ball though now moving in a smaller circle than previously still attains the same height.

Repeat the experiment again, driving the nail in however at H a point between A and E, nearer to A than to E. Then the ball after the string has become vertical will describe a circle of radius HA about H. But this circle will not cut the horizontal line CD, the ball cannot rise to the same height as previously, it will be found that the ball completes the whole semicircle HK; its motion after passing through the point K will depend on the position of H and the radius of the circle. The ball may continue to describe the circle winding the string upon the nail, or the string may become slack for a time and the path of the ball alter.

Again, by reversing the direction of motion we may allow the ball to describe first the smaller circle with F as centre, then, when the string becomes vertical contact with F ceases, and the ball proceeds to move about A in the larger circle; it will in this case be found to rise to C, the same height as previously, and this will be the case for all positions of F which will permit the ball to start from some point in the horizontal line CD and describe an arc of a circle about F.

Thus in all these cases the velocity with which the ball starts up the circle AC must be the same. This velocity is acquired by sliding down the various circles corresponding to the different positions of F. Thus the velocity acquired by sliding down any of these circles from points in the horizontal line CD is the same.

Again, take a piece of wood[1], cut into the form of a curve

which no allowance has been made. It is possible to prove this and to make an allowance if required by experimenting with balls of the same size but of different material. With the apparatus as described, however, the correction will be very small.

[1] Instead of using wood the curve may be made out of a strip of sheet metal bent to the required form.

as shewn at FK, Fig. 75, and place it so that the string after passing the vertical comes into contact with the curve. The ball will no longer describe a circle but some curve, as shewn in Fig. 75, depending on the shape of the wood. It will be found however that it still comes to rest at the point G in which its path cuts the horizontal line through C the starting point, or that conversely, if it be started from G it will rise to C.

Thus the velocity acquired in sliding from rest under gravity a given vertical height down any curve is the same.

***117. Velocity on a curve.** The foregoing experiments enable us to find an expression for the velocity acquired in sliding down a curve; for we have seen that this is the same whatever be the form of the curve provided only that the height through which the body moves is constant. Now in Section 116 it has been shewn that in sliding down a smooth plane the velocity v acquired is given by the equation

$$v^2 = 2gh.$$

Thus in sliding from rest through a vertical height h down any curve a particle acquires a velocity v given by the equation

$$v^2 = 2gh.$$

The work done in this case is mgh and this is equal to $\frac{1}{2}mv^2$.

If the particle do not start from rest the corresponding equation may be found thus. Let u be the initial velocity and let it be acquired by sliding through a vertical height h'. Then the velocity v is acquired by sliding through a height $h + h'$.

Hence we have $$v^2 = 2g(h + h'),$$
$$u^2 = 2gh'.$$

Thus subtracting
$$v^2 - u^2 = 2gh.$$

While for the work done we still have
$$U = mgh = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2.$$

This result corresponds exactly to those found in Section 112 for a body falling freely. It can be shewn that it is true for many cases of motion which are actually observed in nature. We may state then as a result of very wide application that when the velocity of a body changes from u to v work has been done on the body and

The work done is equal to $\frac{1}{2}mv^2 - \frac{1}{2}mu^2$.

Hence, **Work** *must be done in order to increase the velocity of a body, and a body in having its velocity decreased can do work.*

Work *is also necessary in many cases to change the position of a body, while in changing its position the body can do work.*

118. Energy. Hence bodies, as we can observe them, have in some circumstances a capacity for doing work on other bodies; by their action momentum is communicated to those other bodies, which are thereby set in motion: work is done. This capacity for doing work is called **Energy**.

DEFINITION. *The* **Energy** *of a body is its capacity for doing work, and is measured by the work which the body can do in changing to some standard state as regards its position and velocity.*

It is sometimes more convenient to measure the energy of a body by the work which must be done on it to bring it to its actual state from some standard condition.

Thus a stone at the edge of a precipice has energy, a touch will send it over the edge to the ground below and in its fall it can do work; we can imagine it connected with another stone just lighter than itself by a fine string passing over a pulley; as it falls it can draw this other stone up. There are of course numberless other ways in which it could do work.

A body then has energy when raised above the Earth. For such a body it is usual to take as the standard state referred to in the definition that in which the body is at rest on the ground. *A body resting on the ground is said to have no energy.* When at a height h it has energy measured by the work done to lift it to that height; this, if the mass be m, is mgh.

DEFINITION. *The energy of the body which depends on its position and not on its velocity is called* **Potential Energy**.

Hence the potential energy of a mass m at a height h is mgh.

A moving body can do work in being stopped. It has energy in consequence of its motion. This form of energy is called **Kinetic Energy.**

Definition. *The energy of a body which depends on its motion is called* **Kinetic Energy.**

The kinetic energy of a body is measured by the work which it can do in being brought to rest.

Proposition 30. *To find the kinetic energy of a body which is moving with uniform velocity and is brought to rest by a uniform force.*

Let m be the mass of the body, v its velocity, F the force, s the distance the body will move before being stopped.

Then the work done in stopping the body is Fs.

Now we have

$$F = ma, \quad \text{and} \quad as = \tfrac{1}{2}v^2.$$

Thus

$$\text{Work} = Fs = mas = \tfrac{1}{2}mv^2.$$

Hence the kinetic energy of the body is $\frac{1}{2}mv^2$.

We may shew that this is a proper measure for the kinetic energy of the body in any case, and not merely when the retardation is constant.

Thus we have

$$\text{Kinetic Energy} = \tfrac{1}{2}mv^2 = \tfrac{1}{2}mv \times v$$

$= \frac{1}{2}$ the product of the momentum and the velocity.

To prove this for a variable force we treat the force as uniform for very short intervals of time, but variable at the end of each interval. Let F_1, F_2 ... be the values of the force and let s_1, s_2 ... be the short distances traversed while the force has the values F_1, F_2 ... etc. respectively. Then when s_1 etc. are very short the work actually done is

$$F_1s_1 + F_2s_2 + \ldots\ldots$$

Now let v_1, v_2 be the velocities at the beginnings of the spaces s_1, s_2 etc. Then during each of these spaces we may treat the retardation as uniform.

Hence
$$F_1 s_1 = \tfrac{1}{2}mv_1^2 - \tfrac{1}{2}mv_2^2,$$
$$F_2 s_2 = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_3^2,$$
$$F_n s_n = \tfrac{1}{2}mv_n^2 - 0$$
if the body comes to rest at the end of the nth space s_n.

Thus adding these terms together $F_1s_1 + F_2s_2 + \dots F_ns_n = \tfrac{1}{2}mv_1^2$.

Hence writing v for v_1 as the velocity of the particle we see that the work done in stopping the body is $\tfrac{1}{2}mv^2$. Thus $\tfrac{1}{2}mv^2$ is the kinetic energy.

119. Change of Form of Energy. The energy of a body may change its form from potential to kinetic and vice versa. Thus a stone at the top of a cliff is at rest but has potential energy mgh; just before it strikes the ground it is moving with velocity v and has kinetic energy $\tfrac{1}{2}mv^2$, but in this position it has no potential energy; we notice however that since the velocity v has been acquired in falling a distance h, we have $mgh = \tfrac{1}{2}mv^2$. Hence, *The kinetic energy at the bottom is equal to the potential energy at the top.* We shall find this result to be of the greatest importance: for the present let us consider some other transformations of energy. A bullet shot upwards from a gun starts with kinetic energy but with no potential energy; as it rises its kinetic energy decreases, for its velocity diminishes, but its potential energy increases, for its height becomes greater; when at the top of its flight it is instantaneously at rest; its kinetic energy is zero. The height to which it rises is found by dividing the energy with which it starts by its weight, for if v be the velocity of projection, K the kinetic energy at start, and h the height the bullet reaches, then
$$mgh = \tfrac{1}{2}mv^2 = K.$$
Hence
$$h = \frac{K}{mg} = \frac{\text{kinetic energy}}{\text{weight of bullet}};$$
as the bullet falls the potential energy becomes again transformed into kinetic energy.

A pendulum[1] bob when at the extremity of its swing has potential energy: if we take the equilibrium position as the standard one from which to measure, and if h be the height of the starting point above this position, then the potential energy is mgh. As the bob moves down to its equilibrium position its potential energy is diminished, its kinetic energy

[1] A simple pendulum is a ball at the end of a string.

increased until when at the lowest point the energy is all kinetic and is $\frac{1}{2}mv^2$; moreover the kinetic energy in this position is equal to the potential energy at starting, for $\frac{1}{2}mv^2 = mgh$. As the pendulum passes this equilibrium position and rises again, the transformation of energy takes place in the other direction, the kinetic energy becomes potential; we see moreover that it remains unchanged in amount since the pendulum rises to the height from which it started.

Many other examples of the transformation of energy might be given; we should find in all the same law, potential energy can be transformed into kinetic or kinetic into potential, but the gain of one is equal to the loss of the other. We will give a formal proof of this statement for one or two cases.

PROPOSITION 31. *To shew that the energy of a body falling freely remains unchanged during the fall.*

Let a body of mass m fall from a point A, Fig. 76, at a height h above the ground. Let v be its velocity when at the point P at a depth z below A, and E its total energy in this position. Then since PB is $h-z$ the height of the body is $h-z$.

Fig. 76.

Hence its potential energy is $mg(h-z)$.

Its velocity is v; hence its kinetic energy is $\frac{1}{2}mv^2$.

Thus $$E = \frac{1}{2}mv^2 + mg(h-z).$$

But the velocity v is acquired by a fall through the distance z.

Therefore $$v^2 = 2gz,$$

and $$\frac{1}{2}mv^2 = mgz.$$

Hence $$E = mgz + mg(h-z)$$
$$= mgh.$$

Thus the energy in any position is mgh, which is equal to the potential energy at the start: the energy remains unchanged in amount during the motion.

We may put the proof slightly differently thus. In falling a depth z the body loses potential energy mgz, it gains kinetic energy $\frac{1}{2}mv^2$, and since $v^2=2gz$ these two are equal; thus the total energy does not change.

The same result follows if the body be projected down with a velocity u, instead of being dropped.

After it has dropped a depth z its total energy E is given as before by

$$E=\tfrac{1}{2}mv^2+mg(h-z).$$

But $$v^2=u^2+2gz.$$

Thus $$E=\tfrac{1}{2}mu^2+mgh,$$

and this is the sum of the kinetic and potential energies at starting.

The same result is true when a body slides down a smooth curve; for in this case the same formulæ hold, h and z being the vertical distances between the various positions of the body.

***120. Mutual Energy.** We have thus arrived at the result that in a large number of cases of motion the energy of the moving body remains constant though it alters in form.

In the cases with which we have been dealing the energy depends on the position of the body relatively to the earth. We have determined the energy on the assumption that the Earth is at rest so far as the motion of the falling body is concerned and that the body falls to it; strictly of course this is not true, the Earth moves towards the body and the body towards the Earth, their accelerations being inversely as their masses; if we allow for this we find that it is the sum of the energies of the Earth and the body which remain constant. We ought not to speak of the potential energy of the body but of the mutual potential energy of the body and the Earth; in the fall some of this energy becomes transformed into the kinetic energies of the body and the Earth; the sum of these two is equal to the loss of mutual potential energy.

We are thus to look upon **Energy** as *a quantity which we can measure and which in such cases of motion as we have been considering remains unchanged during the motion.*

121. Forms of Energy. There are however other cases of motion in which energy apparently disappears. A falling stone just before reaching the ground has energy $\frac{1}{2}mv^2$, after striking the ground it is reduced to rest and has neither kinetic nor potential energy. Two masses which impinge

directly with equal momenta and adhere have kinetic energy before impact, they are reduced to rest and apparently lose this kinetic energy by the impact.

A body which is allowed to slide down a rough surface has potential energy at starting but it is soon brought to rest in a lower position; a railway train in motion has a large supply of kinetic energy; when the brakes are applied and the train stopped this kinetic energy has been dissipated.

Now it can be shewn that in all these cases the energy has merely changed its form. Heat has been produced and it is found that the heat produced is proportional to the energy which has disappeared. The experiments of Joule and others have proved this; the visible kinetic energy of the moving bodies has been changed into the invisible energy of the molecules of those bodies. When the stone falls and strikes the ground, the total energy of the earth and stone remains unchanged; when the two bodies impinge and are brought to rest, they are heated by the impact and the heat is energy equal in amount to the kinetic energy of the masses.

Energy may take other forms besides the potential and kinetic energy of bodies sufficiently large for us to see. The total amount of energy existing in two or more bodies cannot be altered by any mutual action between those bodies[1].

122. Conservation of Energy. It was said above that in many cases of motion the sum of the kinetic and potential energies of the system considered remains the same; we have now been led to a wider generalization as the result of observation and experiment. We may in Maxwell's words state it thus.

PRINCIPLE OF THE CONSERVATION OF ENERGY. *The total energy of any material system is a quantity which can neither be increased nor diminished by any action between the parts of the system though it may be transformed into any of the forms of which energy is susceptible.*

123. Conservation of Energy in Mechanics. When stated as above the principle of the conservation of

[1] For an account of some of the experiments necessary to prove the statements made above, see Glazebrook, *Heat*, Chaps. I. and XIII.

energy is too wide for effective use in **Mechanics**, it includes the whole of **Physical Science.** We can however limit it and put it into a form which will be of assistance to us. Now we have seen that there are some cases of motion in which there is no change in the sum of the kinetic and potential energies, while in other cases energy is dissipated as heat or in some other form. In the first case the system considered is said to be a **Conservative System**; when dealing in Mechanics with a conservative system the two forms of energy with which we are concerned are kinetic and potential; the sum of these two forms is always the same. The gain of kinetic energy in any change is equal to the loss of potential energy during the same change, and vice versa.

DEFINITION. *In Mechanics a system is said to be* **Conservative** *when the amount of work necessary to bring it from any one condition to any other is always the same and does not depend upon the steps by which that change is carried out.*

For example, the same amount of work is necessary to raise a body from one given position to another given position, by whatever path the body be raised, provided that we are concerned only with the mutual action between the Earth and the body, and the constraints introduced by smooth surfaces.

This system then is a conservative system. It can in fact be shewn that if the impressed force or rate of change of momentum of each part of the system depends only on the position of that part relative to the other parts and not on its velocity, then the system is conservative.

A body sliding down a rough surface is losing momentum owing to friction at a rate which depends partly on its speed and on the direction in which it is going; when *sliding down*, it gains momentum from the action of the Earth but loses it owing to friction; when *sliding up*, both actions contribute to the loss of momentum; this system is not conservative.

In order that the principle of the conservation of energy may be of use to us in solving mechanical problems—in which we deal only with the kinetic and potential energies of visible bodies—it is necessary that the system considered should be a conservative one.

In a Conservative system the sum of the kinetic and potential energies of the system can only be changed by action exercised on the system from without.

Assuming then this principle as established by reasoning of a general character from the fundamental laws and definitions, it may be applied to the solution of individual problems.

Thus the system consisting of the Earth and a heavy body moving on a smooth surface is a conservative one, the potential energy depends only on the height z above the surface of the Earth; the kinetic energy is $\frac{1}{2}mv^2$; if the body start from rest at a height h we have

$$\tfrac{1}{2}mv^2 + mgz = mgh,$$

or

$$v^2 = 2g(h - z).$$

Again, the mutual potential energy of two masses m, m_1, for which Newton's law of gravitation holds, can be proved to be mm_1/r, where r is their distance apart. Thus if v, v_1 be their velocities, their total energy is

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}mv_1^2 + mm_1/r.$$

If the bodies move to a distance r' apart at which they have velocities v', v_1', then the energy is

$$\tfrac{1}{2}mv'^2 + \tfrac{1}{2}m_1v_1'^2 + mm_1/r',$$

and these two expressions for the energy are equal.

Thus

$$\tfrac{1}{2}m(v^2 - v'^2) + \tfrac{1}{2}m_1(v_1^2 - v_1'^2) = \frac{mm_1}{r'} - \frac{mm_1}{r}.$$

124. Unit of Energy. Since energy is measured as work the unit of work is the **Unit of Energy**, its actual value then will depend on the units we use for length, time and mass. On the C.G.S. absolute system the unit of energy is the **Erg**; if we are measuring force in grammes' weight, the unit of energy is the **Centimetre-gramme.**

If again we are working in feet and pounds we have on the absolute system as unit of energy the **Foot-poundal**, and in gravitation units the **Foot-pound.**

When however it is stated that the kinetic energy of a moving mass is $\frac{1}{2}mv^2$, the truth of the relation $F = ma$ has

been assumed; this implies that on the C.G.S. system the force is measured in Dynes, the energy therefore is in Centimetre-Dynes or Ergs, while on the F.P.S. system the energy is in Foot-poundals. *In the various statements made in the preceding sections, it is of course assumed that a consistent system of units is employed throughout.*

Thus on the C.G.S. system the statement that the kinetic energy is $\frac{1}{2}mv^2$ means that it is $\frac{1}{2}mv^2$ ergs; if we wish to express it in centimetre-grammes we must remember that 1 dyne is $1/g$ (or $\frac{1}{981}$) of the weight of a gramme.

Hence in centimetre-grammes the kinetic energy is $\frac{1}{2}mv^2/g$ or $\frac{1}{2}mv^2/981$.

Again the kinetic energy of a mass of m pounds moving with a velocity of v feet per second is $\frac{1}{2}mv^2$ foot-poundals or $\frac{1}{2}mv^2/g$ or $\frac{1}{2}mv^2/32{\cdot}2$ foot-pounds.

125. Energy, Momentum and Force. We have thus been led to deal with two quantities,—Energy and Momentum,—depending on the motion of bodies: each of these is unchanged in total amount by mutual action between the bodies which make up the system, each can be transferred from one body of the system to another. Energy may exist in various forms, it may change from one form to the other in the course of the motion, but all these forms can be measured in terms of one common unit, and when so measured their sum total remains the same. Momentum we only know in the form of the product of the mass of a body and its velocity.

Momentum and kinetic energy are closely connected; to measure the kinetic energy of a particle we multiply its momentum by its velocity and divide by 2.

Force stands in a different category to these two quantities, its amount does not in general remain unchanged during the motion; we find however that the conception of force is useful to connect together momentum and energy, and to express certain observed facts.

Thus suppose a body of mass m is observed to move from rest with a uniform acceleration a, and to describe a distance s in time t. Then we find that the following three quantities —mv/t, ma and $\frac{1}{2}mv^2/s$—each of which we can determine by observation are equal.

Each of these may be defined to be the impressed force: calling it F we have

$$F = \frac{mv}{t} = ma = \frac{\frac{1}{2}mv^2}{s}.$$

If the initial velocity be u the formulæ become

$$F = \frac{m(v-u)}{t} = ma = \frac{\frac{1}{2}mv^2 - \frac{1}{2}mu^2}{s}.$$

We also deal with the products Ft which measures the whole change of momentum or the Impulse, and Fs which measures the Work or whole increase of kinetic energy.

We find moreover that in a large number of cases these quantities depend only on the position of the body with reference to surrounding objects and are quite independent of its velocity or direction of motion. To each of these quantities the name of Force is given. These results are obtained from the observation of certain simple cases of motion. They are then generalized and by their aid the motion of bodies under very complex circumstances can be calculated.

Newton founded the Science of Dynamics on the first two of the above relations. Force he defines as rate of change of momentum. In the Corollary to the third law he draws attention to the importance of the last relation and emphasizes some of its principles. In his view it follows as a mathematical consequence of the first relation; it might of course have been taken as the starting point of the subject, basing it as has been done in the preceding pages on the fact that a body moving along a smooth curve will rise to the same height as that from which it started; this indeed had been done by Galileo, and it was by this method that Huyghens had obtained his results about the motion of pendulums.

***126. Graphical construction for Work.** When the impressed force is uniform the work done is Fs; when it is variable we suppose it to be uniform while the body moves over a number of very short spaces s_1, s_2 etc., but to change at the end of each of these spaces. We can express this by a graphical construction identical with that used in Section 41 to determine the space traversed in terms of the time.

PROPOSITION 32. *To determine graphically the work done by a force.*

Draw a line OX, Fig. 77, to represent space and let NN' be the distance traversed under a force F. Draw NP and $N'P'$ at right angles to NN' to represent the force and join PP'. Then the area $PNN'P'$ is equal to Fs. But Fs is the work done on the body when moving the distance s; hence the area $PNN'P'$ represents the work.

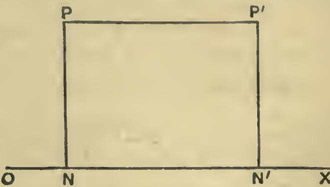


Fig. 77.

If the force be not constant but change at the ends of the spaces s_1, s_2... etc., from F_1 to F_2, F_2 to F_3, etc., respectively, the work done will be represented by the area consisting of a number of rectangles such as $N_1P_1R_1N_2$, $N_2P_2R_2N_3$, etc., Fig. 78.

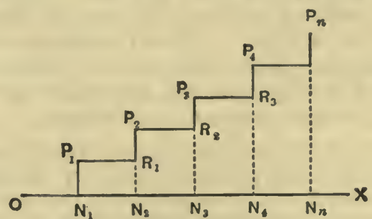


Fig. 78.

By diminishing the spaces N_1N_2, N_2N_3, etc., during which we deal with the force as uniform, we get the case of a varying force; the broken line $P_1R_1P_2$... of Fig. 78 becomes a

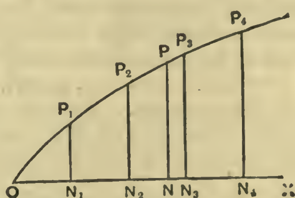


Fig. 79.

continuous curve P_1PP_4, Fig. 79, and the area $P_1N_1N_4P_4$ represents the work done by traversing the distance N_1N_4. Thus if we construct a figure on squared paper in which the horizontal divisions represent space, and the vertical divisions force, by drawing a curve P_1P' such that the line PN perpendicular to the space line may represent the force when the body has traversed a space N_1N, the area $P_1N_1N'P'$ represents the work done during the motion. In such a diagram if the horizontal divisions represent centimetres and the vertical divisions be dynes, then the work in ergs is given by the number of squares contained within the area.

The following is an example of the method.

Proposition 33. *To calculate the work done in stretching a spiral spring.*

Let ON_1, Fig. 80, be the original length of the spring and suppose it stretched so that its length is ON. Then N_1N represents the extension. Now we have seen, Section 91, that the force required to extend a spring is proportional to the extension. Thus if PN drawn vertical at N represent the force which will just hold the spring extended to the length ON, we see that PN is always proportional to N_1N: if we wish to extend the spring by twice N_1N the force necessary to hold it in this position will be twice PN. Thus the curve corresponding, Fig. 79, which gives the force in terms of the displacement, is a straight line through the points P and N_1, and the area which determines the work is a triangle.

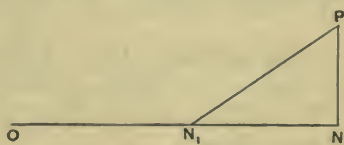


Fig. 80.

Thus the work done in extending the spring from N_1 to N is the area of the triangle PNN_1.

Hence if we call F the force, and s the final extension produced under this force, we see that

$$\text{Work done} = \text{Area } PNN_1$$
$$= \tfrac{1}{2}PN \, . \, NN_1 = \tfrac{1}{2}Fs.$$

Examples. (1). *Find the energy of a mass of* 1 *cwt. while falling from a height of* 100 *feet.*

The total energy at any point of the fall is equal to the potential energy at the starting-point, and this is equal to the work done in raising the body from the ground. This work has been shewn Example 1, p. 160, to be 15190 Joules.

(2). *Compare* (*a*) *the momenta,* (*b*) *the kinetic energies of a bullet whose mass is* 100 *grammes moving with a speed of* 400 *metres per second, and a cannon-ball whose mass is* 50 *kilogrammes moving with a speed of* 10 *metres per second.*

Reduce the speeds to centimetres per second and the masses to grammes, then we have

(*a*) Momentum of the bullet $= 100 \times 400 \times 100$
$= 4 \times 10^6$ C.G.S. units of momentum.

Momentum of cannon-ball $= 50 \times 1000 \times 10 \times 100$
$= 5 \times 10^7$ C.G.S. units of momentum.

(*b*) Energy of bullet $= \frac{1}{2} 100 \times 16 \times 10^8$
$= 8 \times 10^{10}$ ergs.

Energy of cannon-ball $= \frac{1}{2} 50 \times 1000 \times 1 \times 10^6$
$= 2{\cdot}5 \times 10^{10}$ ergs.

Thus the cannon-ball has the greater momentum while the bullet has the greater energy.

(3). *The bullet and the cannon-ball are each brought to rest with uniform retardation in* 1 *second. Determine the impressed forces and the distance each moves.*

In the case of the bullet 4×10^6 units of momentum are destroyed in one second, thus the impressed force is 4×10^6 dynes and the retardation is $4 \times 10^6/100$ or 4000 cm. per sec. per sec.

The distance the bullet will travel while being stopped in one second is thus $\frac{1}{2}$ (4000) or 2000 centimetres.

For the cannon-ball 5×10^7 units of momentum are destroyed in one second, thus the impressed force is 5×10^7 dynes, and since the mass is 50000 grammes the retardation is $5 \times 10^7/5 \times 10^4$ or 1000 cm. per sec. per sec. The distance the cannon-ball moves while being stopped in 1 second is thus 500 centimetres.

Thus if both bodies are stopped in the same time and both lose their momentum at uniform rates, the rate of loss for the cannon-ball is $\frac{50}{4}$ or 12·5 times as great as that for the bullet, but the bullet moves $\frac{20}{5}$ or 4 times as far as the cannon-ball.

(4). *A bullet* 100 *grammes in mass is fired from a gun the barrel of which is* 75 *cm. long and leaves it with a velocity of* 400 *metres per second; assuming the pressure due to the powder to be uniform, find the impressed force on the bullet and the time it takes to traverse the barrel.*

Let F dynes be the impressed force. The kinetic energy of the bullet is $\frac{1}{2}\,100 \times (40000)^2$, or 8×10^{10} ergs.

It acquires this energy while moving through 75 centimetres. Hence

$$F \times 75 = 8 \times 10^{10}.$$

$$F = \frac{8 \times 4 \times 10^{10}}{3 \times 10^2} = \frac{32}{3} \times 10^8 \text{ dynes.}$$

The momentum of the bullet is 4×10^6 C. G. S. units. The time during which the bullet is in the barrel is given by dividing this by the impressed force; let it be t seconds.

Then $$t = \frac{4 \times 10^6 \times 3}{32 \times 10^8} = \frac{3}{8} \times 10^{-2},$$

or three eight-hundredths of a second.

(5). *An engine developes* 5000 *horse-power while driving a ship at the rate of* 25 *miles an hour. Find the resistance offered to the motion.*

The energy supplied by the engine is employed in doing work against the resistance. The velocity of the ship is $36\frac{2}{3}$ feet per second, the rate at which work is being done is 550×5000 foot-pounds per second, and this is equal to the resistance multiplied by the velocity. Hence if R represent the resistance in lb. weight

$$R \times 36\tfrac{2}{3} = 550 \times 5000,$$

$$R = \frac{550 \times 5000 \times 3}{110} = 75000 \text{ lb. weight.}$$

(6). *A simple pendulum, the mass of which is* 1 *kilogramme, is started from its lowest point with a velocity of* 120 *cm. per second; the pendulum makes one oscillation per second and loses energy from friction and other causes at the rate of* 1 *centimetre-gramme unit per second. Determine for how long it will continue to move.*

The kinetic energy of the pendulum is

$$\tfrac{1}{2} \times 1000 \times 14400 \text{ ergs,}$$

or $$\tfrac{1}{2} \times \frac{1000 \times 14400}{981} \text{ centimetre-gramme units.}$$

This reduces to 7339 centimetre-gramme units of energy. Since 1 of these units is lost each second, the pendulum will continue to move for 7339 seconds or for 2 hours 2 minutes 19 seconds.

EXAMPLES.

WORK AND ENERGY.

1. Define Energy, and explain how to measure (*a*) the energy of a bullet as it leaves the muzzle of a gun, (*b*) the energy of a clock pendulum at the highest and lowest points of its swing.

2. State the principle of the Conservation of Energy as employed in Mechanics, and illustrate it by some examples.

By the use of this principle shew that the velocity acquired by a body falling from rest down a smooth inclined plane depends only upon the vertical height of the plane and not upon its length.

3. Distinguish between work and power. A watt is equivalent to 10^7 ergs per second; the acceleration of gravity is 981 cm. per sec. per sec.; find how long a kilogramme has been falling from rest when gravity is doing work upon it at the rate of one watt.

4. Calculate the momentum and the energy of (1) a bullet weighing $\frac{1}{2}$ an oz. moving at the rate of 1200 feet per second, (2) a mass of $\frac{1}{2}$ a ton moving at the rate of 6 inches per second. Find the forces required to stop the two in $\frac{1}{10}$ second and the work which each would do in being stopped.

5. If a body be moving under the action of a constant force, shew that the horse-power developed by the force is proportional to the force and to the velocity of the body.

6. Distinguish between kinetic and potential energy.

A pendulum consisting of a ten-gramme bob at the end of a string thirty centimetres long oscillates through a semi-circle; find its kinetic energy when the string makes an angle of 45° with the vertical.

Specify the units in which your answer is expressed.

7. Shew how the second law of motion enables us to measure force and mass. What do you understand by "action" in the statement of the third law? Illustrate your answer by some applications of the law.

8. A mass of 50 grammes moving with a velocity of 12 cm. per second overtakes and adheres to a mass of 30 grammes moving with a velocity of 4 cm. per second. Find the common velocity and calculate the total kinetic energy before and after the impact.

9. A mass of 1 cwt. is moving with a velocity of 1 foot per second. Determine the velocity of a bullet whose mass is 1 oz. when it has (1) the same momentum, and (2) the same kinetic energy as the larger mass.

10. A force equal to the weight of 10 lb. acts for a minute on a mass of 1 cwt. Find the momentum and energy of the mass. What is the work done by the force?

11. A bullet whose mass is 1 oz. leaves the muzzle of a gun with the velocity of 1000 feet per second, find its energy in foot-poundals; and if the length of the barrel of the gun is 3 feet, find the mean pressure exerted by the powder on the bullet.

12. A bullet whose mass is an ounce moving with a velocity of 2400 feet per second strikes a block of wood at rest and remains imbedded in the wood: if the resulting velocity of the block and bullet together be 16 feet per second, calculate the mass of the wood. Also calculate the energy lost when the bullet penetrates the wood, and express the result in foot-tons.

13. A man whose mass is 150 lb. walks up a hill of 1 in 6 at the rate of 4 miles an hour; what fraction of a horse-power is he doing?

14. Find the amount of work done in pushing a mass of 10 lb. through 5 feet up an incline of 1 in 10, neglecting friction.

15. Find the amount of horse-power transmitted by a rope passing over a wheel 10 feet in diameter which makes 1 revolution per second, the tension in the rope being 100 lbs.

16. Point out the transformations of energy that take place during the swinging of a pendulum. State at what point of its swing a pendulum must be if its energy is half potential and half kinetic.

17. How much energy has a mass of 1 cwt. when moving at the rate of 100 yds. per sec.? In what units is your answer expressed?

18. A man can bicycle 12 miles an hour on a smooth road. He exerts a downward pressure of 20 lb.-wt. with each foot during the down-stroke, and the length of down-stroke is 12 inches. His driving wheel is 12 feet in circumference. Find the work he does per minute.

19. A shot whose mass is half a ton is fired with a velocity of 2000 feet a second from a gun whose mass is 50 tons; neglecting the weight of the powder, find the velocity with which the gun will recoil, if mounted so that it moves without friction along a level tramway.

Compare the work done on the gun with that done on the shot.

20. A cannon weighs 35 tons, and the shot half a ton. The velocity of the shot on leaving the muzzle is 1200 ft. per second; find the velocity of the recoil of the cannon. Neglect the inertia of the gases formed by the burning of the powder.

Will the effect of these gases be to reduce or to increase the recoil? Give your reasons.

21. What is the horse-power required to fill in 3 hours a tank 9 feet deep, 20 feet long, and 10 feet wide, placed on the top of a building 60 feet in height, from a well in which the surface of the water remains constantly 24 feet below the ground level? Give the answer correct to two places of decimals.

(Mass of cubic foot of water $= 62\frac{1}{2}$ lb.)

22. Assuming that the resistance of a train moving along a horizontal railway at 35 miles per hour is equivalent to an incline of 1 in 280 find the horse-power required to take a train of 100 tons along a horizontal railway at a rate of 35 miles per hour.

23. The mass of a ship is 3000 tons; assuming that the resistance to its motion varies as the square of the speed and that the force required to give it a speed of 1 foot per second is equal to $\frac{1}{150000}$ the weight of the ship, find the horse-power requisite to propel it at the rate of 30 feet per second.

24. Define the kinetic energy and the potential energy of a system.

A fine string passes through two small fixed rings, A and B in the same horizontal plane, and carries equal weights at its ends, hanging freely from A and B. If a third equal weight is attached to the middle point of the portion AB of the string, and is let go, prove that it will descend to a depth equal to two-thirds of the length AB below AB, and will then ascend again.

25. A mass of 4 cwt. falls from a height of 10 feet on to an inelastic pile of 12 cwt. Supposing the mean resistance to the penetration of the pile to be $1\frac{1}{2}$ tons weight, determine the distance it is driven at each blow.

26. A smooth wedge of mass M and angle α rests upon a horizontal plane; another mass m is placed upon its slant surface, and the system begins to move.

Write down (1) the equation of energy, (2) the equation of linear momentum.

27. I strike an anvil with a given hammer, and its velocity on reaching the anvil is the same as that with which it reaches a piece of red-hot iron on the anvil. Will the impulse be the same in the two cases?

What quantities must be known in order to compare impulses?

28. A bullet weighing 1 oz. leaves the mouth of a rifle whose barrel is 4 feet long with the velocity of 1000 ft. per second. Find the mean force on the bullet, neglecting the friction against the sides of the barrel.

29. A cannon-ball whose mass is 1 cwt. moving with a velocity of 25 yds. per sec. penetrates to a depth of 10 feet into a sandbank. Find the average pressure on the sand.

30. The erg and foot-pound are both units of work: a horse-power is 33,000 foot-pounds per minute; how many ergs per hour would this be?

[1 inch = 2·54 centimetres. 1 lb. = 453·6 grammes.]

31. In a system of distributing power by means of water at a high pressure, the pressure of water is 2000 lb.-wt. per square inch. How many cubic feet must be used per hour to supply 10 H.-P. (1 H.-P. = 33000 foot-pounds of work per minute) assuming no power to be lost?

32. A mass of m pounds is raised up a plane inclined at an angle of 30° to the horizon, and of length l, and reaches the top with a velocity v. Shew that the work done is

$$\frac{m}{2}\left(1+\frac{v^2}{g}\right) \text{ foot-pounds.}$$

*CHAPTER IX.

CURVILINEAR MOTION UNDER GRAVITY.

127. Projectiles. When a ball is thrown in the air it does not move in a straight line; a very little observation is sufficient to shew this; moreover its velocity is continually changing. We can deduce the form of the path from the laws of motion, thus—Let us suppose, in order to simplify the problem, that the body is projected in a horizontal direction with a velocity u; the only acceleration which it acquires is g in a vertical direction due to the action of the Earth, and it is not difficult to determine the path of the body under these circumstances. We will however in the first instance investigate the motion by the aid of experiment.

We have seen that two bodies whatever be their masses when dropped together fall at the same rate.

We wish now to shew that, if one of the bodies be projected in a horizontal direction with any velocity while the other is allowed to drop simultaneously from the same height, the two will fall at the same rate and reach the ground together. We may shew this roughly by rolling a ball rapidly along a table; on leaving the table it will describe a curve in the air; if now at the moment the first ball leaves the table a second be dropped from the same height the two will reach the ground together. The same fact is better shewn by the aid of an arrangement of apparatus devised by Sir Robert Ball[1].

[1] *Experimental Mechanics*, Section 511.

Experiment 24. *To shew that the time of fall of a body is independent of its horizontal velocity.*

In Fig. 81, *AB* is a piece of wood about 2·5 cm. thick the upper edge of which is curved as shewn. A strip of thin

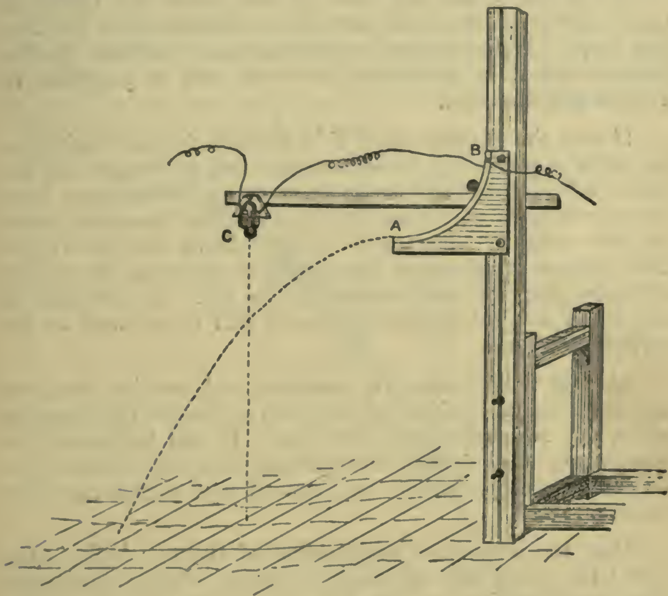


Fig. 81.

brass is screwed to each face of the board, the edges of the brass strips being also curved: care must be taken that there shall be no metallic connection between the brass strips. Thus the upper edge of the wooden board forms a kind of groove with brass sides. The brass strips are connected to binding screws. On resting a brass ball on the strips as shewn in the figure, an electric current can pass from one strip to the other through the ball.

One binding screw is connected to a battery, the other to the electromagnet described in Section 65; a wire also

passes from the electro-magnet to the second pole of the battery. When the ball is on the groove the circuit is complete and the magnet is made; thus it can support a small iron ball.

When the brass ball rolls off the groove the circuit is broken between the strips and at the same moment the iron ball drops. If the groove be fixed in such a position that its direction at A is horizontal the brass ball is projected in a horizontal direction.

If also the electromagnet C be fixed at the same height as the point of projection A, then the iron ball is dropped and the brass ball projected horizontally at the same moment from the same height. The velocity with which the brass ball starts can be varied by allowing it to roll[1] down the groove AB from various positions, or by placing a spring in the groove and projecting the ball forward by its aid. If this plan be adopted a straight horizontal groove will do as well as the curved one shewn.

Arrange the apparatus as described and, starting the brass ball from various points in the groove, observe the times at which the two balls reach the floor. It will be found that whatever be the height of the starting point and whatever be the velocity of projection the two always strike the floor simultaneously.

Thus the downward acceleration of the two balls is the same; the brass ball although it starts with a horizontal velocity, depending on the distance it has rolled down the groove, gains in each second the same vertical velocity as the iron ball; in the first second it will fall through $\frac{1}{2}g$ centimetres, in the second through $\frac{4}{2}g$ centimetres, and in t seconds $\frac{1}{2}gt^2$ centimetres. The vertical velocity is independent of the horizontal and is the same as that of a body allowed to drop freely.

[1] If the ball is not quite spherical then in rolling down it may happen that contact between the ball and the strip is broken, and the iron ball is allowed to drop too soon. This may be avoided by using instead of a ball a cylindrical piece of brass or zinc which slides down on the brass strips; the friction however in this case is greater and the ball does not start with so great a velocity.

EXPERIMENT 25. *To describe a parabola.*

Take a straight lath or scale about a metre long; divide it into equal distances each 10 cm. in length and from each point of division suspend by a piece of fine string or thread a small bullet or weight.

Adjust the first piece of string to some convenient length, say 3 centimetres, make the second 3×2^2 cm., the third 3×3^2 cm., the fourth 3×4^2 and so on, so that the lengths of any two strings are proportional to the squares of their distances from the end of the rod.

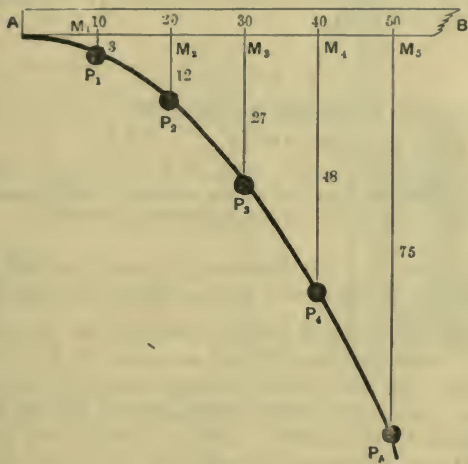


Fig. 82.

Thus, in Fig. 82, AB is the lath, M_1, M_2, M_3, etc. the points of division, P_1, P_2 etc. the bullets, and we have $P_1M_1 = 3$ cm.,

$$P_2M_2 = 3 \times 2^2 = 12 \text{ cm.},$$

$$P_3M_3 = 3 \times 3^2 = 27 \text{ cm.},$$

and so on.

Hold the lath as shewn in Fig. 82 against a vertical black-board or sheet of drawing-paper, mark the positions P_1, P_2 etc.

of the bullets. The points so found lie on a curve called a parabola. If we suppose that each of the 10-centimetre divisions is subdivided into (say) 10 parts, and that threads with bullets are hung from these in such a way that the lengths of the threads may follow the same law as before, the bullets will be practically in contact and the curve which passes through them will be a parabola. The curve however can be constructed with sufficient accuracy for our purposes by drawing a free-hand curve through the points $P_1P_2 \dots$

Again, draw a vertical line AN as in Fig. 83 from A, the end of the lath from which the divisions are reckoned, and from P_1 draw P_1N_1 parallel to the lath to meet AN in N_1.

Then
$$P_1N_1 = AM_1,$$
$$AN_1 = PM_1,$$

and by construction PM_1 is proportional to AM^2.

Hence $P_1N_1^2$ is proportional to AN_1.

In the figure the lath is horizontal, this however is not necessary; in whatever direction the lath be held the balls still lie on a parabola; the size of the curve will depend on the inclination of the lath.

EXPERIMENT 26. *To determine the path of a body projected in a horizontal direction and to shew that it is a parabola.*

Arrange the grooved board described in Experiment 24 in front of a vertical black-board as shewn in Fig. 83 in such a way that the ball after sliding down the groove may start from A in a horizontal direction and fall in a vertical plane parallel to the board. Make a mark C on the groove and in all the experiments start the ball from this mark. Allow the ball to roll down the groove and watch its path.

Fix to the board with drawing-pins a number of paper hoops so that the ball in its path passes through each of them; the proper position of the uppermost hoop is first found by trial, then that of the next below, and so on, the ball being started in each case from the same point C. Mark on the board the positions of the centres of the hoops, remove them and draw with a free hand a curve starting from A and passing through the various marks P_1, P_2 .... Draw a horizontal line from A, and vertical lines P_1M_1, P_2M_2 ... from the marks

to meet it in M_1, M_2. Measure the horizontal distances AM_1, AM_2..., and the vertical distances P_1M_1, P_2M_2, etc.

Write down the squares of AM_1, AM_2 etc., and obtain the quotients, given by dividing each square such as AM_1^2, by

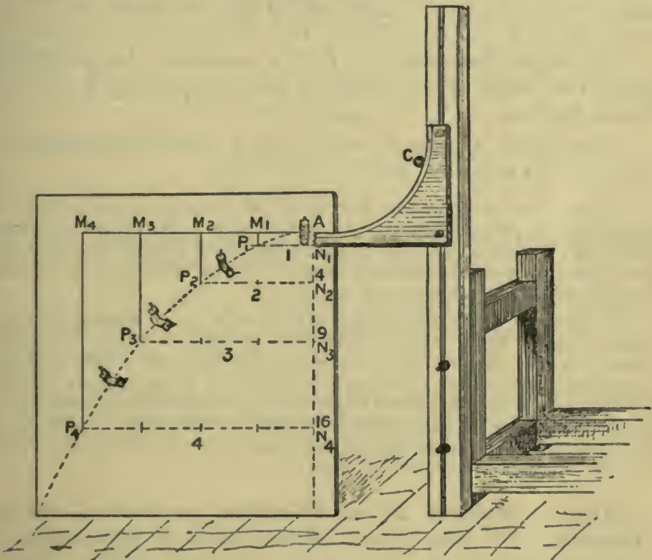


Fig. 83.

the corresponding vertical distance; it will be found that these quotients are all approximately equal. But the curve which has this property is the parabola. Thus the path is a parabola.

Again, measure the vertical height above A of the point C from which the ball starts; let it be a. Then it will be found that the constant ratio of AM^2 to PM is equal to $4a$.

Thus we have

$$AM^2 = 4aPM.$$

But if u be the horizontal velocity with which the ball leaves the groove we have seen that u is acquired by falling down a height a, hence $u^2 = 2ga$.

Therefore $$a = u^2/2g,$$

and $$AM^2 = \frac{2u^2}{g} PM.$$

The curve described by P is a parabola, the point A is called the vertex of the parabola, and the quantity $2u^2/g$ is its Latus Rectum.

Thus, when a body is projected in a horizontal direction its path is a **Parabola.**

Again, if we suppose the motion of the body at any point to be reversed, it will proceed to describe the same path in the reverse direction; thus when projected obliquely its path will still be a parabola. This may be verified by construction in a similar way by arranging the grooved board so that its direction is not horizontal.

The path of the drops of water in a water-jet is the same as that of a body projected under gravity, each little particle of water follows the same course as that which would be taken by the body if projected with the velocity with which it starts. By placing a lamp at some distance from such a jet its shadow can be thrown on a white screen placed behind it, the path of the jet can thus be traced and measurements made on it as on the curve drawn as described above. For further details see Glazebrook and Shaw, *Practical Physics*, Section C.

128. Motion of a Particle projected under Gravity. We have shewn by the result of Experiment 24 that the vertical motion of a falling body is independent of its horizontal velocity. A body which has initially no vertical velocity will in t seconds fall a distance $\frac{1}{2}gt^2$, whether it start from rest or be projected horizontally.

If the body be projected obliquely, its velocity has a vertical as well as a horizontal component; let v be the upward vertical component, u the horizontal component.

Then during t seconds the velocity v will have carried the body a distance vt upwards, while owing to the vertical acceleration g the displacement will be $\frac{1}{2}gt^2$ downwards. Thus h, the actual height above the point of projection, is given by

$$h = vt - \frac{1}{2}gt^2.$$

In the same time the horizontal velocity u will have carried it a distance ut, and it has no horizontal acceleration, hence k the horizontal displacement is given by

$$k = ut.$$

If the actual velocity of projection be U and the direction of motion be inclined at an angle a to the horizon, then we have

$$u = U \cos a,$$
$$v = U \sin a.$$

Hence

$$h = Ut \sin a - \tfrac{1}{2}gt^2,$$
$$k = Ut \cos a.$$

We will now shew how to prove that the path is a parabola so long as the resistance of the air is neglected.

Proposition 34. *A particle is projected with a given velocity and at a given inclination to the horizon; to shew that its path is a parabola.*

Let P, Fig. 84, be the point of projection and PT, making an angle a with the horizontal line Px, the direction of projection. Let U be the velocity of projection along PT.

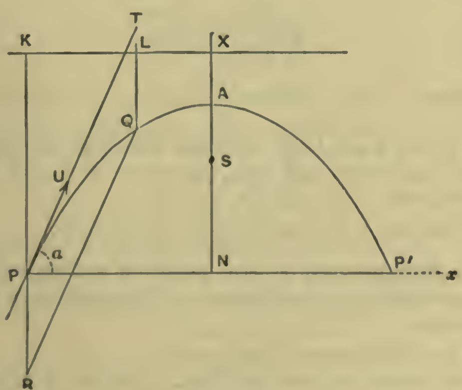


Fig. 84.

The only acceleration which the particle has is vertical and is equal to g. The motion will therefore take place in the vertical plane through PT.

To find the position of the particle after t seconds, make PT equal to Ut and from T draw TQ vertically downwards and equal to $\frac{1}{2}gt^2$. Draw PR vertical and equal to TQ and join QR. Then if the particle had no acceleration it would move uniformly along PT and at the end of t seconds would be at T. Again, if it had initially no velocity it would in t seconds fall vertically and reach the point R. In the actual circumstances the displacement in each of these two directions is, in accordance with the second law, independent of that in the other, the particle is displaced in the direction PT just as far as it would be if it had no vertical acceleration; it is displaced vertically just as far as it would be if it had initially no velocity along PT. Thus, at the end of t seconds, PT still represents the displacement due to the initial velocity, while PR or TQ represent the displacement due to the acceleration. Hence the particle is at Q.

Now
$$QT = \tfrac{1}{2}gt^2,$$
$$PT = Ut.$$

Hence
$$t^2 = \frac{2QT}{g}, \quad t = \frac{PT}{U},$$

and
$$2\frac{QT}{g} = t^2 = \frac{PT^2}{U^2}.$$

Therefore
$$PT^2 = \frac{2U^2}{g} . QT.$$

Now U and g are both constant, therefore the ratio of PT^2 to QT is a constant, and this (Experiment 25) is the fundamental property of a parabola.

Thus the point Q always lies on a parabola.

129. Properties of the path of a Projectile. We will now proceed to investigate various properties of the motion of a projectile.

(1) *To find the directrix of the parabola.*

Draw PK vertical and equal to $U^2/2g$. Then the velocity at P would be acquired by falling through a height KP. A horizontal line KX drawn

through K is called the directrix. The velocity at any point of the parabola is that due to the fall from the directrix.

For if PK is equal to H, and if h be the height above P of the particle when at Q, since the energy remains constant we have, if V denote the velocity at Q,

$$\tfrac{1}{2}mV^2 + mgh = \tfrac{1}{2}mU^2 = mgH.$$

Hence $V^2 = 2g(H-h) = 2gQL$ if the vertical QT meet the directrix in L.

(2) *To find the vertical and horizontal components of the velocity at any time.*

The horizontal velocity initially is $U\cos\alpha$, and since there is no horizontal acceleration it remains unchanged.

The vertical velocity initially is $U\sin\alpha$, and in t seconds under the downward vertical acceleration g an additional vertical velocity $-gt$ is acquired.

Hence if u, v represent the components of the velocity at any time

$$u = U\cos\alpha,$$
$$v = U\sin\alpha - gt.$$

(3) *To find the direction of motion at any time.*

If at any time t the particle be moving with velocity V in a direction making an angle θ with the horizon, we have

$$V\cos\theta = u = U\cos\alpha,$$
$$V\sin\theta = v = U\sin\alpha - gt.$$

Hence
$$V^2 = U^2\cos^2\alpha + (U\sin\alpha - gt)^2$$
$$= U^2 + g^2t^2 - 2Ugt\sin\alpha,$$
$$\tan\theta = \frac{U\sin\alpha - gt}{U\cos\alpha}.$$

(4) *To find the position of the particle at any time.*

Let h be the distance of the particle above P, k its horizontal distance from P.

Then
$$h = Ut\sin\alpha - \tfrac{1}{2}gt^2,$$
$$k = Ut\cos\alpha.$$

(5) *To find the time to the vertex.*

At the vertex A (Fig. 84) the motion is horizontal, the vertical velocity therefore is zero. Hence if t_1 is the time to the vertex

$$U\sin\alpha - gt_1 = 0,$$

$$t_1 = \frac{U\sin\alpha}{g}.$$

(6) *To find the height of the vertex.*

If the height of the vertex AN (Fig. 84) be h_1, then h_1 is the height of the particle at time t_1 when the vertical velocity is zero.

Hence $$h_1 = Ut_1 \sin \alpha - \tfrac{1}{2} g t_1^2.$$

But $$t_1 = \frac{U \sin \alpha}{g}.$$

Therefore $$h_1 = \tfrac{1}{2} \frac{U^2 \sin^2 \alpha}{g}.$$

(7) *To find the distance of the vertex from the directrix.*

At the vertex the velocity is horizontal and is equal to $u \cos \alpha$.

Hence if AX be perpendicular from A on the vertex, we have

$$AX = \frac{U^2 \cos^2 \alpha}{2g}.$$

In a parabola four times the distance between the vertex and the directrix is the latus rectum. Hence the latus rectum is $2U^2 \cos^2 \alpha / g$.

In Section 125 we found the value $2u^2/g$ for the latus rectum; it must be remembered that u is the constant horizontal velocity which is equal to $U \cos \alpha$. Thus the two formulæ are the same.

A line through the vertex at right angles to the directrix is called the axis of the parabola.

A point S on this line at a distance from the vertex equal to AX is called the focus.

(8) *To find the horizontal distance between the vertex and the point of projection.*

Let the distance PN (Fig. 84) be k_1. Then k_1 represents the horizontal distance which the particle has moved in time t_1.

Hence $$k_1 = U \cos \alpha t_1 = \frac{U^2 \cos \alpha \sin \alpha}{g} = u \frac{U \sin \alpha}{g},$$

writing u for the constant horizontal velocity $U \cos \alpha$.

Moreover $$h_1 = \frac{U^2 \sin^2 \alpha}{2g}.$$

Hence $$k_1^2 = \frac{2u^2}{g} \frac{U^2 \sin^2 \alpha}{2g}$$
$$= \frac{2u^2}{g} h_1.$$

Thus $$PN^2 = 4AX \, . \, AN,$$

and the ratio of PN^2 to AN is the same for all points. (Cf. Section 125.)

(9) *To find the time at which the particle reaches the ground again.*

Let the time be t_2.

Then at time t_2 the height h of the particle is zero.

Hence $$0 = Ut_2 \sin \alpha - \tfrac{1}{2} g t_2^2.$$

Therefore $t_2 = 0$, which gives the starting-point, or

$$t_2 = \frac{2U \sin \alpha}{g} = 2t_1,$$

which gives the time to the point P'.

Thus the particle takes as long to descend from A to P' as to rise from P to A.

The value of t_2 is known as the time of flight.

(10) *To find the range on the horizontal plane.*

The range PP' is the horizontal distance which the particle moves in time t_2.

Hence $$\text{Range} = Ut_2 \cos \alpha = \frac{2U^2 \sin \alpha \cos \alpha}{g}$$

$$= \frac{2u}{g} . U \sin \alpha = \frac{U^2 \sin 2\alpha}{g}.$$

Hence for a given velocity of projection the range is greatest if $\sin 2\alpha$ is greatest, that is if $2\alpha = 90°$, $\alpha = 45°$.

A number of other propositions on parabolic motion might be given; for these the reader is referred to Loney, *Elementary Dynamics*.

Examples. (**1**). *Shew by finding an expression for the velocity of a projectile that it is equal to that due to a fall from the directrix.*

If H be the distance between the directrix and the point of projection, h the height of the particle at time t above the point of projection.

We have $$U^2 = 2gH, \quad h = Ut \sin \alpha - \tfrac{1}{2} g t^2.$$

$$\begin{aligned} V^2 &= U^2 \cos^2 \alpha + (U \sin \alpha - gt)^2 \\ &= U^2 + g^2 t^2 - 2gtU \sin \alpha \\ &= U^2 - 2g\,(Ut \sin \alpha - \tfrac{1}{2} g t^2) \\ &= U^2 - 2gh \\ &= 2g\,(H - h). \end{aligned}$$

Thus V is the velocity due to a fall from the directrix.

(**2**). *Find the time at which a particle projected with velocity U in direction α will strike a plane through the point of projection inclined at an angle β.*

We have with the same notation

$$\begin{aligned} h &= Ut \sin \alpha - \tfrac{1}{2} g t^2, \\ k &= Ut \cos \alpha, \\ h &= k \tan \beta. \end{aligned}$$

Hence $$U \sin\alpha - \tfrac{1}{2}gt = U\cos\alpha\tan\beta.$$

Therefore $$t = \frac{2U}{g}(\sin\alpha - \cos\alpha\tan\beta)$$
$$= \frac{2U}{g}\frac{\sin(\alpha-\beta)}{\cos\beta}.$$

(3). *Find the angle at which a particle must be projected so as to hit a given point if the velocity of projection be given.*

Let h, k be the vertical and horizontal distances of the point from the point of projection and t the time to the point.

Then $$t = \frac{k}{U\cos\alpha},$$
$$h = Ut\sin\alpha - \tfrac{1}{2}gt^2$$
$$= k\tan\alpha - \tfrac{1}{2}g\frac{k^2}{U^2\cos^2\alpha}$$
$$= k\tan\alpha - \tfrac{1}{2}\frac{gk^2}{U^2}(1+\tan^2\alpha).$$

This gives us a quadratic equation to find α, and corresponding to the two roots we have two directions of projection.

(4). *A stone is thrown from a cliff* 112 *feet high with a velocity of* 192 *feet per second in a direction making an angle of* 30° *with the horizon; find where it strikes the ground.*

The vertical velocity is 192 sin 30 or 96 feet per second, the horizontal velocity is $96\sqrt{3}$ feet per second. When projected up with a velocity of 96 feet per second it will be at a distance of 112 feet *below* its point of projection after a time T given by

$$-112 = 96T - \tfrac{1}{2}gT^2$$
$$= 96T - 16T^2,$$
$$T^2 - 6T - 7 = 0.$$

Hence $$T = 7, \text{ or } T = -1.$$

Thus the stone will strike the ground 7 seconds after it started; we also see that the stone might have been projected from the ground with proper velocity 1 second before it started from the cliff; it would then have passed the edge of the cliff at the moment of starting with a velocity of 192 feet per second at an inclination of 30° to the horizon.

Since the horizontal velocity of the stone is $96\sqrt{3}$ feet per second, the horizontal distance from the cliff of the point at which it strikes the ground after 7 seconds will be $7\times 96\sqrt{3}$ feet.

(5). *A man can just throw a stone 392 feet. Find the velocity with which he throws it; find also how high it will rise and determine the time of flight.*

The range is greatest when the angle of projection is 45°, and then the range is U^2/g.

Hence $$\frac{U^2}{g} = 392 \text{ feet}.$$

Hence $$U^2 = 32 \times 392,$$

whence $$U = 112 \text{ feet per second}.$$

The greatest height to which it rises is

$$\tfrac{1}{2} U^2 \sin^2 a/g, \text{ and } \sin^2 a = \tfrac{1}{2}.$$?

Hence greatest height is $32 \times 392/4 \times 32$, or 98 feet.

The time of flight ($2U \sin a/g$) is $112\sqrt{2}/32$, or $7/\sqrt{2}$ seconds

130. The Simple Pendulum. A heavy particle suspended by a fine flexible string constitutes a simple pendulum. In practice we cannot of course arrange that the string should be perfectly flexible or that the body which is suspended should be a particle. For most purposes however a spherical ball of wood or metal suspended by a piece of fine string such as a waterproofed fishing-line will serve as a simple pendulum; the ball may be conveniently from 5 to 7 cm. in diameter.

If such a pendulum be drawn aside and then let go, it commences to oscillate backwards and forwards in a vertical plane. The bob of the pendulum moves in an arc of a circle. The distance from the point of suspension to the centre of the suspended sphere is called the length of the pendulum, and we

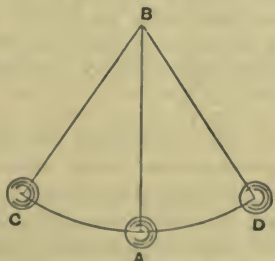


Fig. 85.

treat the motion as though the bob were a heavy particle concentrated at its centre. Such a pendulum is shewn in Fig. 85. The length of the arc AC through which the pendulum swings, measured from its lowest position, is known as the **Amplitude of the vibration.** Such a pendulum when once started loses its energy very slowly and will continue to swing for a long time; its amplitude gradually grows less but the decrease is very slow.

Now Galileo shewed that, if the amplitude of oscillation of a pendulum be not large, its time of swing is constant; thus if at starting it takes the pendulum 1 second to move from C through A to D and back to C, it will continue throughout its motion to take 1 second for each such oscillation. The uniform rate of a clock depends on this property of a pendulum. In order to verify it completely we should need to start a long heavy pendulum and count the number of oscillations in the interval between some two astronomical occurrences, which are always separated by a constant interval of time, such as the transit across the meridian of two known stars. If we find that during any such interval the pendulum makes a number of oscillations which is proportional to the length of the interval, we infer that the duration of each oscillation is a constant number of seconds. We thus arrive at the result that in a given locality the time of swing of a given pendulum is independent of the amplitude. But we can shew more than this, for we find also that the time of swing does not depend on the mass of the bob. The bob may be of any material, provided only the length of the pendulum remains unchanged, and the conditions such that we may treat the motion as that of a simple pendulum, the time of swing is the same. Newton called attention to this and made numerous experiments to verify the fact.

EXPERIMENT 27. *To shew that the time of swing of a simple pendulum is independent of the mass of the bob.*

Take a number of spheres of about the same size, but of different materials, and suspend them all side by side from some steady support, such as a horizontal bar, by strings of the same length (say 1 metre). This is most easily done by having an eye screwed into each sphere through which the string can

pass. The string is passed through a small hole in a small wooden block, it is then threaded through the eye of the bob and the end is secured to another hole in the same small block. The length of the string can then be adjusted by sliding the block up and down in the same way as the stay-ropes of a tent are tightened, and the friction will hold the block in any position. Adjust the strings of each pendulum carefully till all are of the same length, then start the pendulums swinging simultaneously. To do this, place a board against the spheres and push them all aside to the same extent. On withdrawing the board, the pendulums all start together and will continue if their lengths have been carefully adjusted to keep time for a large number of oscillations; thus the time of swing is independent of the mass of the bob.

If this experiment be continued for some time it will be found that the lighter spheres begin to lag behind. This, as Newton shewed, is due to the resistance of the air; if the experiment were performed in a vacuum no such effect would be noticed, the effect of the resistance of the air may be allowed for by observing the decrease that takes place in the amplitude of successive swings. When this is done it is found that the mass of the bob does not affect the time of swing.

131. Relation between Weight and Mass. This result affords a more accurate verification of the law that the weight of a body in a given locality is proportional to its mass than can be obtained from observations on a falling body—a pendulum is practically a falling body whose motion we can observe for a long period of time.

Consider now two of the pendulums. At any moment their velocities and accelerations are respectively the same. Their masses however are different, the force acting in each case is the same definite fraction of the weight of either pendulum, a fraction which depends on the inclination of the pendulum string to the vertical at that moment. Since the acceleration is the same the ratio of the force acting on each pendulum to the mass of the pendulum is the same for the two; hence the ratio of the weight of the pendulum to its mass is the same for all the pendulums, but this ratio measures g, the acceleration due to gravity, thus this quantity is the same for any two bodies.

132. Experiments with Pendulums.

EXPERIMENT 28. *To shew that the time of swing of a pendulum varies as the square root of its length.*

Fit up side by side three pendulums; let the length of one be l cm. (l may conveniently be about 35 cm.), that of the next 2^2l or $4l$ cm., and that of the third 3^2l or $9l$ cm. Start the first two vibrating simultaneously. Count the number of oscillations made by the shorter one while the longer one makes some 6 or 8, it will be found to be double the number made by the longer; for each oscillation of the long pendulum the short one makes two. The observation is most easily made by placing one hand so that the short pendulum at the extremity of each swing may just come up to it without contact, while the other hand is held in a similar position with regard to the long pendulum. It will then be found that in a given period the short pendulum approaches the one hand twice as often as the long pendulum approaches the other; the time of vibration of the long pendulum is twice that of the short, but the length of the long pendulum is four times that of the short, and two is the square root of four, hence in this case the times of swing are proportional to the square roots of the lengths. Now make similar observations with the first and third pendulums; it will be found in this case that the short pendulum makes three oscillations while the long one makes one, the times are as 1 to 3 while the lengths are as 1^2 to 3^2, thus the times are proportional to the square roots of the lengths. If a number of simple pendulums of different lengths be made to vibrate, the time of swing of each can be observed with a stop-watch, and the length of each measured. Make these observations for each pendulum, then form a table in which one column contains the observed times while the other contains the square roots of the lengths, it will be found that the corresponding entries in the two columns are proportional.

EXPERIMENT 29. *To verify the formula that in a simple pendulum* $t = 2\pi\sqrt{\frac{l}{g}}$, *where* t *is the time of a complete oscillation in seconds,* l *the length of the pendulum in centimetres, and* g *the acceleration of a falling body in centimetres per second per second.*

One end of a thin string is fastened to a fixed support, the other is passed through a ring attached to a heavy ball and then fastened to a small piece of wood sliding on the string. In this way a simple pendulum of adjustable length is obtained. Make the pendulum about 100 cm. long and measure carefully the distance from the point of support to the centre of the ball. Place some mark, such as a vertical rod standing on the floor, to indicate the position of the pendulum at the lowest point. Start the pendulum swinging and determine with a watch the number of seconds taken by the ball to pass 51 times in the same direction through its lowest point; in reckoning the transits it is best to count the first transit as 0, the fifty-first will then reckon as 50, and the number of seconds which has elapsed will be the time of 50 vibrations. Divide this by 50, we have the time of an oscillation, let this be t seconds. If a stop-watch is used it should be started at the first transit and stopped at the fifty-first.

Now substitute in the formula the measured length of the pendulum l and the value of g (in England 981 cm. per second per second) and thus compute the value of the quantity $2\pi\sqrt{\frac{l}{g}}$; it will be found that this quantity is equal to t: alter the length of the pendulum, making it twice as long as before, and make another similar series of observations, it will be found that the time t is altered in the ratio of $\sqrt{2}$ to 1 or approximately 1·41 to 1 and the formula is again verified.

133. Period of Oscillation of a Pendulum. The formula verified by the preceding experiment can be deduced from the laws of motion when applied to the case of a simple pendulum: if we assume the formula true the same observations give us a means of determining g, for we can observe t and l, and then we have

$$t = 2\pi\sqrt{\frac{l}{g}}.$$

Thus $$g = \frac{4\pi^2 l}{t^2},$$

and from this we can calculate g.

This is the method used to determine g with accuracy; in practice however a simple pendulum is not employed, a metal bar is made to oscillate about an axis perpendicular to its length; a formula can be obtained connecting together the time of swing, the dimensions of the bar and the value of g, and from this g can be found[1], the other quantities being known.

It is beyond our limits to prove the formula here. We may indicate how it is obtained in the following way.

Consider a particle falling down an inclined plane CA (Fig. 86). From C draw CD at right angles to the plane, and from A draw AD vertical to meet CD in D. Bisect AD in B, then a circle with B as centre will pass through D, C and A; let l be its radius, and let a be the angle which AC makes with the horizontal line AT which touches the circle at A. Let t_1 be the time taken to slide from C to A; the acceleration down the plane is $g \sin a$, hence

$$AC = \tfrac{1}{2} g \sin a t_1^2.$$

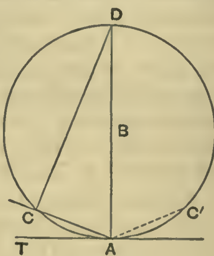


Fig. 86.

But from the figure the angle ADC is a, and $AC = AD \sin ADC = 2l \sin a$.

Thus $$2l \sin a = \tfrac{1}{2} g \sin a t_1^2.$$

Hence $$t_1 = 2\sqrt{\frac{l}{g}}.$$

Now if we imagine a similar plane AC' on the opposite side of AD to AC, and that the particle could be started up this with the velocity it has at the bottom, it would rise to a height equal to that from which it started, and come to rest after a second interval of t_1 seconds, it would then descend and rise to C after another interval of $2t_1$ seconds; the whole time of an oscillation then would be $4t_1$, and we should have

$$\text{Time of an oscillation} = 4t_1 = 8\sqrt{\frac{l}{g}}.$$

Now if the plane is short and the angle a is small there is not much difference between the plane and the small circular arc AC, along which a particle attached to a string of length l at B would move. As a rough approximation therefore to the time of an oscillation in such a small circular arc we have the value $8\sqrt{l/g}$. Such an approximation however is only rough, for the particle starts at C along a steeper slope than the plane; its acceleration at first therefore is greater than on the plane, and

[1] See Glazebrook and Shaw, *Practical Physics*, § 20, for details of this and of the method of determining t with accuracy.

this gives it an additional velocity which more than compensates for the less acceleration of the arc near C. When allowance is made for the varying inclination of the arc, it is found that we must replace the 8 of the above formula by 2π or 6·28 approximately, and then we get

$$t = 2\pi\sqrt{\frac{l}{g}}.$$

The rough formula however illustrates the method of proof, and shews how the value $\sqrt{l/g}$ comes in. (See Section 146, Example.)

134. Value of g in different latitudes. Attention has already been called to the fact that while g is a constant for all bodies at a given point on the earth's surface, it varies from point to point. Pendulum experiments afford the best means of establishing this. If the time of oscillation of the same pendulum be observed in different latitudes it is found to vary, thus a pendulum of given length oscillates more quickly at the pole than at the equator; since the value of g is inversely proportional to the square of the time of an oscillation we can compare the values of g by comparing the square of the times of an oscillation. It was in great measure a knowledge of the fact, derived from such experiments, that g was variable which led Newton to distinguish between mass and weight.

Examples. (**1**). *Find the length of a pendulum which makes* 1 *complete oscillation per second at a place where the value of* g *is* 981 *cm. per sec. per sec.*

If the length of the pendulum be l cm.

then $$1 = 2\pi\sqrt{\frac{l}{981}}.$$

Hence $$l = \frac{981}{4\pi^2} = 24{\cdot}84 \text{ cm.}$$

A "seconds pendulum," as the phrase is generally employed, means one which passes through its equilibrium position once a second. Its time of swing therefore is 2 seconds and its length is four times the above.

Hence Length of a seconds pendulum under the above conditions

$$= 99{\cdot}36 \text{ cm.}$$

The actual value of g in London is 981·17, and the length of the seconds pendulum is 99·413 cm.

(2). *The value of* g *at the equator is* 978·1, *in London* 981·17, *and at the pole* 983·11; *the length of the seconds pendulum is* 99·413 *cm. in London; find its value at the pole and the equator.*

Since the time of swing is to be constant, the lengths are proportional to the values of g.

Hence

$$\text{Length at Equator} = \frac{978 \cdot 1}{981 \cdot 17} \times 99 \cdot 413 = 99 \cdot 103 \text{ cm.}$$

$$\text{Length at Pole} = \frac{983 \cdot 11}{981 \cdot 17} \times 99 \cdot 413 = 99 \cdot 610 \text{ cm.}$$

(3). *A pendulum beats seconds at London; find with the above data its time of swing at the equator.*

The length remains the same; thus the periods are inversely proportional to the square roots of the value of g.

Hence

$$\text{Period at Equator} = \sqrt{\frac{981 \cdot 17}{978 \cdot 10}}$$

$$= 1 \cdot 00156 \text{ seconds.}$$

EXAMPLES.

PROJECTILES.

1. Determine the velocity with which a stone must be projected at an angle of 45° to the horizon in order that the range may be 100 yards.

2. A stone is projected with a velocity of 50 feet per second in a direction making an angle θ with the horizon, where $\tan\theta = \frac{3}{4}$. Find the greatest height it attains.

3. If v is the vertical component of the velocity of projection of a particle, prove that the greatest height it attains above the horizontal plane through the starting-point is $\frac{v^2}{2g}$.

4. A stone is projected with a velocity of 60 feet per second in a direction making an angle θ with the horizon, where $\tan\theta = \frac{4}{3}$. Find its range on a horizontal plane through the starting-point, and the time of flight.

5. A cannon-ball is observed to strike the surface of the sea, to rebound, and to strike the surface again 2000 yards further on after 6 seconds. Find the horizontal velocity of the shot during the rebound and the greatest height the shot attains.

6. From the top of a vertical tower, whose height above the horizontal plane on which it stands is $\frac{13}{2}g$ feet, a heavy particle is projected with a velocity whose upward vertical and horizontal components are $6g$ and $8g$ feet per second respectively; find the time of flight and the distance from the base of the tower at which the particle will strike the ground, g being the acceleration due to gravity.

7. A particle is projected with a velocity V in a direction making an angle α with the horizon, under the action of gravity; find the highest point to which the particle will rise and the range on the horizontal plane passing through the point. If the velocity of projection be 880 feet per second, find in miles the greatest range on the plane, supposing the acceleration due to gravity to be 32 feet per second.

8. A shot is fired horizontally from the top of a tower with a velocity equal to the vertical component of its velocity when it reaches the ground; shew that it reaches the ground at a distance from the foot of the tower equal to twice the height of the tower.

9. A particle is projected with a velocity u in a direction making an angle θ with the horizontal. Find the range on the horizontal plane through the point of projection.

If the range is 300 feet, and the time of flight 5 seconds, find the velocity of projection.

10. A particle is projected with a velocity u in a direction making an angle θ with the vertical. Find the greatest height to which it will rise above the horizontal plane through the point of projection.

If the greatest height is 49 feet and the velocity at the highest point is 42 feet per second, find the velocity of projection.

11. If a particle be projected horizontally, with a given velocity u, along the surface of a smooth plane, inclined at an angle α to the horizontal, what will be the latus rectum of its path?

12. A bullet is fired from a gun at an elevation of 45°, with an initial velocity of 840 feet per second. Find the range on a horizontal plane through the point of projection.

If the bullet strikes a bird which rose vertically with uniform velocity from a point on the ground 500 yards distant from the gun at the instant when the shot was fired, find the velocity of the bird.

13. Prove that the range of a projectile on a horizontal plane through the point of projection is $2uv/g$, where u and v are the horizontal and vertical components of the velocity of projection, and g is the acceleration due to gravity.

14. A particle is projected with velocity V at an angle α with the horizon; find the height of the focus of the path.

15. A ball rolls from rest at the top of the roof of a house and finally strikes the ground at a distance from the house equal to its breadth. If the roof on both sides of the house makes an angle of 30° with the horizon, prove that the height of the house to the eaves of the roof is three times the sloping distance from the eaves to the top of the roof. [The top of the roof is in the middle of the house.]

16. Shew that if lines be drawn from a point to represent the velocities of a projectile at different instants their other extremities will lie on a vertical line.

17. A vertical line is divided into a number of equal parts A_1A_2, A_2A_3, A_3A_4 etc. Shew that if a particle be projected from O in the vertical plane through the line, OA_1, OA_2, OA_3 etc. will meet its path in points such that the times of flight from each to the next are all the same.

18. Find the range of a projectile on a horizontal plane passing through the point of projection; and prove that when the velocity of projection has a given value u there are two possible directions of projection such that the range has a given value R; provided R is less than a certain distance.

19. Prove that in this case the difference of the greatest heights attained in the two paths is $\frac{1}{2}\sqrt{\frac{u^4}{g^2} - R^2}$.

*CHAPTER X.

COLLISION.

135. Impact. We have seen already that experiment proves that when two bodies impinge there is neither loss nor gain of momentum, and it has been pointed out that in order to calculate the motion when the two impinging bodies do not adhere some further experimental result is necessary.

Newton's experiments already referred to afford the necessary information. He proved by measuring with the aid of the apparatus shewn in Fig. 51, Section 50, that when two spheres impinge directly their relative velocity after impact always bears a fixed ratio to that before impact; thus, if u, u' are the velocities of the spheres before impact and v, v' after impact, all estimated in the same direction, the relative velocities are $u-u'$ and $v-v'$ respectively. Now with Newton's apparatus the velocities can be measured and it is shewn as the result of experiments that the ratio of $v-v'$ to $u-u'$ is always the same for balls of the same two materials; it does not depend on the masses of the balls but only on the substances of which they are composed.

This constant ratio is found to be a negative fraction, that is to say if u is greater than u', so that $u-u'$ is positive, then v is less than v' or $v-v'$ is negative, the ball which is struck has the greater velocity after impact; the ratio is never greater than unity; in some cases the balls separate with the same relative velocity as that with which they impinged though the

direction of this velocity is changed; in general however the relative velocity is reduced by the impact. If we denote the ratio of $v - v'$ to $u - u'$ by $-e$, then the quantity e is never greater than unity: it is called the **Coefficient of Restitution.** For certain pairs of substances the coefficient of restitution is unity.

In general we have

$$\frac{v - v'}{u - u'} = -e.$$

We can now make use of these two results of experiment to solve some problems on impact.

PROPOSITION 35. *Two balls impinge directly; to find their velocities after impact in terms of the velocities before impact, their masses, and the coefficient of restitution.*

Let m, m' be the masses of the two balls A, B, Fig. 87,

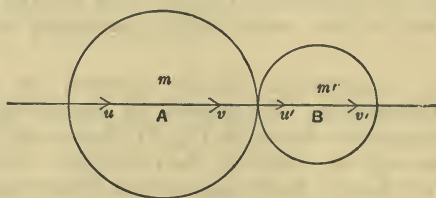


Fig. 87.

u, u' their velocities before impact, v, v' their velocities after impact, and e the coefficient of restitution.

We suppose that initially the two balls are moving in the same direction AB, the velocity (u) of A being greater than that of B.

We know that the momentum is unchanged by the impact and that the ratio of the relative velocity after impact to that before impact is $-e$.

Thus we have

$$mv + m'v' = mu + m'u',$$

$$v - v' = -e(u - u') = -eu + eu'.$$

Hence, solving these equations

$$(m+m')\,v=(m-em')\,u+m'(1+e)\,u',$$
$$(m+m')\,v'=m(1+e)\,u+(m'-em)\,u'.$$

These two equations give the values of v and v' when u and u' are known; they are simplified if e is unity, in which case the balls are said to be perfectly elastic, or if e is zero, in which case the balls adhere and move with the common velocity $(mu+m'u')/(m+m')$. In the latter case with which we have already dealt at length the balls are said to be inelastic. (See Section 58.)

PROPOSITION 36. *To find the Impulse of an Impact.*

Let the Impulse of the motion be I. Then we have

$$I=mv-mu=-(m'v'-m'u').$$

Now subtracting $(m+m')\,u$ from both sides of the equation which gives v we have

$$(m+m')\,(v-u)=(m-em')\,u-(m+m')\,u+m'(1+e)\,u'.$$

Hence $$v-u=\frac{m'(1+e)}{m+m'}(u'-u).$$

Therefore $$I=\frac{mm'(1+e)}{m+m'}(u'-u).$$

PROPOSITION 37. *To find the velocity of rebound after direct impact on a fixed surface.*

We may deal with this problem by supposing the mass of the second ball to be very large and its initial velocity to be zero. It will remain at rest: we must put m' infinite and u' zero in the equations and we get

$$v=-eu, \quad v'=0.$$

Or we may obtain this from Newton's second experimental result, the relative velocity before impact is u, after impact it is v, and we have $v=-eu$.

We cannot use the first experimental result, for though u' is zero, m' is very large, and we do not know what the value of $m'u'$ is.

Examples. In working examples it is much the best plan always to have recourse to the two fundamental principles and not to quote the results for v and v'. Thus

(1). *A mass of* 1 *kilogramme moving with a velocity of* 50 *cm. per second impinges directly on a mass of* 10 *kilogrammes at rest; the coefficient of restitution is* $\frac{1}{2}$. *Find the velocities after impact.*

Let the velocities after impact be v and v' respectively in centimetres per second. The momentum before impact is 50000 gramme-centimetre units, and the relative velocity 50 cm. per sec.

Hence $$1000v + 10000v' = 50000,$$

$$v - v' = -\tfrac{1}{2}50 = -25.$$

Hence $$10v' + v = 50,$$

$$v' - v = 25,$$

$$11v' = 75,$$

$$v' = \tfrac{75}{11} = 6\tfrac{9}{11} \text{ cm. per sec.,}$$

$$v = -(25 - 6\tfrac{9}{11}) = -18\tfrac{2}{11} \text{ cm. per sec.}$$

Thus the 1 kilogramme ball returns with a velocity of $18\frac{2}{11}$ cm. per sec., the 10 kilogramme ball moves forward with a velocity of $6\frac{9}{11}$ cm. per sec.

(2). *A ball whose mass is* 1 *lb. moving with a velocity of* 10 *feet per second impinges directly on another ball moving in the opposite direction with a velocity of* 5 *feet per second. The first ball is observed after impact to continue to move on with a velocity of* 5 *feet per second, and the coefficient of restitution is* $\frac{2}{3}$. *Find the mass and the velocity after impact of the second ball.*

Let the mass be m' lb. and the velocity estimated in the direction in which the 1 lb. ball moves v' ft. per sec. The momentum before impact in this direction is $10 - 5m'$.

The relative velocity before impact is $10 + 5$ or 15.

Hence $$5 + m'v' = 10 - 5m',$$

$$5 - v' = -\tfrac{2}{3}15 = -10.$$

$$\therefore\ v' = 15.$$

Hence $$5 + 15m' = 10 - 5m',$$

or $$20m' = 5,$$

$$m' = \tfrac{1}{4}.$$

Thus the mass of the second ball is $\frac{1}{4}$ of a lb. and it moves after impact in the direction opposite to that of its initial motion with a velocity of 15 feet per second.

(3). *A ball drops from a height of 25 feet on to a horizontal surface and rebounds, the coefficient of restitution is $\frac{3}{4}$; find how high the ball will rise after striking the surface three times.*

Let the velocity with which it strikes the surface be u ft. per sec. Then since the velocity is acquired by falling through 25 feet

$$u = \sqrt{2g \,.\, 25} = \sqrt{64 \times 25} = 40 \text{ feet per sec.}$$

After 1 rebound this becomes $\frac{3}{4}$ of 40, or 30 feet per second.

The ball rises and then falls again, striking the ground with a velocity of 30 feet per second. After this second impact the velocity becomes $\frac{3}{4}$ of 30, and after the third impact it becomes

$$\tfrac{3}{4} \times \tfrac{3}{4} \times 30 \text{ or } \tfrac{9}{8} \times 15.$$

The height to which the ball will rise then is

$$\left(\frac{9 \times 15}{8}\right)^2 \Big/ 2g.$$

Thus the height required is equal to

$$\left(\frac{135}{64}\right)^2, \text{ or about } 4{\cdot}448 \text{ feet.}$$

136. Energy and Impact.

PROPOSITION 38. *To find the work done when two bodies impinge directly.*

The kinetic energy of each ball is changed by the impact; thus work is done, and the work done on either ball is equal to its gain of kinetic energy.

Hence the work done on the ball A

$$\begin{aligned} &= \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2 \\ &= \tfrac{1}{2}m\,(v-u)\,(v+u) \\ &= \tfrac{1}{2}I\,(v+u), \end{aligned}$$

if I is the impulse or whole change of momentum.

The work done on the ball B

$$\begin{aligned} &= \tfrac{1}{2}m'v'^2 - \tfrac{1}{2}m'u'^2 \\ &= \tfrac{1}{2}m'\,(v'-u')\,(v'+u') \\ &= -\tfrac{1}{2}I\,(v'+u'), \end{aligned}$$

for $m'(v'-u') = -m(v-u) = -I$.

PROPOSITION 39. *To find the whole change of kinetic energy on Impact.*

We have just seen that the ball A gains an amount $\frac{1}{2}I(v+u)$ of energy, while the ball B loses an amount

$$\tfrac{1}{2}I(v'+u').$$

Hence the total loss of kinetic energy is

$$\tfrac{1}{2}I(v'+u'-v-u).$$

Now $$v-v'=-e(u-u').$$

Thus the loss of kinetic energy is

$$\tfrac{1}{2}I(u'-u)(1-e).$$

On substituting the value of I from Prop. 38, we find for the loss of kinetic energy

$$\frac{1}{2}\frac{mm'}{m+m'}(1-e^2)(u'-u)^2.$$

Now $(u'-u)^2$ being a square is always positive and e^2 is not greater than unity, hence $1-e^2$ is positive unless $e=1$, when it is zero. Thus the loss of kinetic energy on impact is always positive unless the coefficient of restitution is unity, when there is no loss. In this case kinetic energy is transferred from one ball to the other but its amount is unchanged. In general however kinetic energy disappears. Joule's experiments, already referred to, lead us to believe that the total energy is unchanged, the energy apparently lost in the kinetic form is transformed into heat, the balls are raised in temperature, and the heat needed for this is measured by the loss of kinetic energy.

137. Oblique Impact. In the cases of impact which have been considered it has been supposed that the two balls were moving either in the same or in exactly opposite directions at the point of impact. This is not always the case: consider the two balls A, B, Fig. 88, let their directions of motion make angles α, α' before impact with the line joining their centres, which is known as the line of impact, and β, β' with the same line after impact, their velocities and masses being as in the previous sections. In considering the problem we have to deal

with the motion along the line AB and also with that at right angles to it. Now the whole momentum is unchanged, and

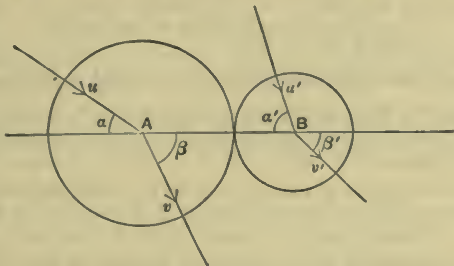


Fig. 88.

this statement is true for the components of the momentum along and perpendicular to the line of impact.

If the balls be smooth there is no force between them at right angles to the line of impact. Thus the velocity of each ball in this direction remains unchanged.

Newton's experimental result as to the relative velocity before and after impact applies to the velocities along the line of impact.

Proposition 40. *To determine the motion after impact of two balls which impinge obliquely.*

The first principle above stated gives us the following results:

$$mv\cos\beta + m'v'\cos\beta' = mu\cos\alpha + m'u'\cos\alpha' \quad\text{......(i)},$$

$$mv\sin\beta + m'v'\sin\beta' = mu\sin\alpha + m'u'\sin\alpha' \quad\text{......(ii)}.$$

From the second we have

$$v\sin\beta = u\sin\alpha \quad\text{....................(iii)},$$

$$v'\sin\beta' = u'\sin\alpha' \quad\text{..................(iv)},$$

while from the third we find that

$$v\cos\beta - v'\cos\beta' = -e\,(u\cos\alpha - u'\cos\alpha') \quad\text{......(v)}.$$

Of these five equations (ii) is included in (iii) and (iv). Hence we have four independent equations (i), (iii), (iv) and (v) from which we can find v, v', β and β'.

We may shew as above that kinetic energy is lost by the impact and that the amount so lost is

$$\frac{1}{2}\frac{mm'}{m+m'}(1-e^2)(u\cos\alpha-u'\cos\alpha')^2.$$

If one of the bodies be fixed we proceed as follows:

Example. *A billiard-ball moving with velocity* u *strikes the cushion at an angle* α, *the coefficient of restitution being* e; *find the direction and velocity of rebound.*

Let v be the velocity of rebound and let the direction of motion after impact make an angle β with the normal to the cushion at the point of impact.

Then the velocity along the cushion is unchanged, that perpendicular to the cushion is reversed and reduced in the ratio e to 1.

Hence if the velocities be estimated as in Fig. 89, we have

$$v\sin\beta=u\sin\alpha,$$
$$v\cos\beta=eu\cos\alpha$$

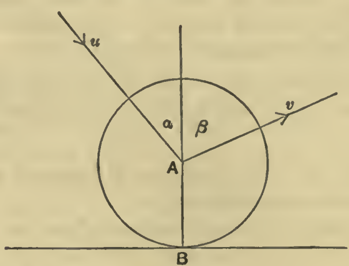


Fig. 89.

Thus $$v^2=u^2(\sin^2\alpha+e^2\cos^2\alpha),$$
$$\cot\beta=e\cot\alpha.$$

If the coefficient of restitution be unity, we have

$$e=1, \text{ then } v=u, \ \beta=\alpha.$$

The ball rebounds with its velocity unchanged in amount, and its direction of motion inclined to the cushion at the same angle as before, but on the opposite side of the normal.

138. Action during Impact. The term Action has been used in our discussion of the third law either for the rate

at which momentum is transferred, or the rate at which work is done. Now in a case of impact the momentum of each ball is changed by a finite amount in a very brief period, the rate of change of momentum is very great, too great for observation; we do not deal with the rate of change of momentum but with the whole change which occurs during the impact. This whole change is what is meant by the Impulse of the motion, so that *when we speak of the* Action *taking place between two bodies at impact we refer to the whole amount of momentum which is transferred.*

139. Moment of greatest compression. It is convenient in some cases to divide the whole change of momentum into two parts. Consider what takes place at the point of impact: the bodies are deformed, the velocity of the one A is being reduced, that of the other B is being increased, there will be a moment during the very brief interval in which the two are in contact at which the deformation of each ball has reached its maximum amount and the two balls are moving together with the same velocity. Let us then divide the duration of the impact into two parts, the first lasting up to this instant of greatest compression, the second, during which the balls are again separating, from this instant up to the time at which contact ceases. Let I_1 be the momentum transferred in the first part, I_2 in the second, then we have

$$I = I_1 + I_2.$$

PROPOSITION 41. *To shew that the impulse after the moment of greatest compression bears a constant ratio to that before, and that this ratio is the coefficient of restitution.*

By hypothesis the two balls have a common velocity at the instant at which an amount I_1 of momentum has been transferred. Let this velocity be V; then

$$m(V-u) = I_1 = -m'(V-u').$$

Hence $$V = \frac{mu + m'u'}{m+m'}.$$

Thus $$I_1 = \frac{mm'}{m+m'}(u'-u).$$

But $$I = \frac{mm'(1+e)}{m+m'}(u'-u).$$

Hence $$I = (1+e) I_1.$$

But $$I = I_1 + I_2.$$

Therefore $$I_2 = eI_1.$$

Hence the impulse during the second period of the impact bears a constant ratio to that during the first part, and this ratio is measured by the coefficient of restitution.

Thus when a ball impinges directly on a flat surface and rebounds, during the first part of the impact all its momentum is transferred to the surface and the ball is reduced to rest, during the second period an amount of momentum e times as great as that which it has lost is acquired by the ball in the opposite direction, it rebounds therefore with a velocity e times as great as that with which it struck the surface.

Example. *A ball whose mass is 1 lb. moving with a velocity of 10 feet per second overtakes another ball whose mass is $\frac{1}{4}$ lb. moving in the same direction with a velocity of 5 feet per second. The coefficient of restitution is $\frac{3}{4}$; find the impulse up to the moment of greatest compression and the whole impulse.*

Let V be the common velocity at the moment of greatest compression. Then since the momentum is unchanged

$$(1+\tfrac{1}{4})V = 1 \times 10 + \tfrac{1}{4} \times 5,$$

$$\frac{5V}{4} = \tfrac{45}{4},$$

$$V = 9 \text{ feet per second.}$$

Thus the Impulse up to this time is $1(9-10)$ or -1.

The ball loses 1 lb.-foot unit of momentum.

The total impulse therefore is $1(1+\frac{3}{4})$ or $\frac{7}{4}$ lb.-foot units of momentum. These are lost by the first ball, gained by the second. Hence if v' be its final velocity we have

$$\tfrac{1}{4}v' = \tfrac{7}{4} + \tfrac{5}{4}.$$

Thus after impact the smaller ball has a velocity of 12 feet per second.

EXAMPLES.

IMPACT.

1. An inelastic ball impinges directly on another of half its mass at rest, find the new velocity of the two in terms of that of the impinging ball.

2. The velocities of two balls before impact are 10 and 6 feet per second respectively, after impact they are 5 and 8 feet per second respectively; compare the masses of the two balls and find the coefficient of restitution.

3. Two bodies of unequal mass moving in opposite directions with equal momenta impinge directly. Shew that their momenta are equal after impact.

4. A body whose mass is 3 lb. impinges directly on one whose mass is 1 lb., the coefficient of restitution is $\frac{1}{3}$. After impact the momenta of the two balls is the same and the smaller has a velocity of 15 feet per second; find the original velocities of the two.

5. Find the condition that two balls may interchange velocities on direct impact.

6. A body is dropped from a height of 64 feet on to a horizontal floor. If the coefficient of restitution be $\frac{1}{3}$, find the height to which the body rises after 3 rebounds.

7. A ball strikes a cushion at an angle of 45°. If the coefficient of restitution be $\frac{3}{4}$, find the angle of rebound.

8. A ball impinges with a velocity u on an equal ball at rest, the direction of motion making an angle of 30° with the line of the centres; determine the motion of the two balls afterwards, the coefficient of restitution being unity.

9. A ball impinges obliquely on an equal ball at rest, find the direction of impact if the two move afterwards with equal velocities, the coefficient of restitution being unity.

10. Two balls of masses m, m', whose coefficient of elasticity is e, impinge directly on each other with velocities u, v respectively ($u > v$). Shew that the impulse between the balls is

$$\frac{mm'}{m+m'}(u-v)(1+e).$$

11. Of three equal balls A, B, C, placed in this order in one straight line, A moves with a given initial velocity towards B, while B and C are at rest. Find B's velocity after it has struck C, assuming the coefficients of elasticity to be the same for the two pairs of balls A, B and B, C.

12. Find the velocities after impact of two smooth spheres which impinge directly, in terms of their masses, their velocities before impact, and the coefficient of restitution.

Shew that the velocity of their centre of inertia is unaltered by the impact, and that the velocity of each body relative to the centre of inertia is reversed in direction and diminished in the ratio of $1:e$, where e is the coefficient of restitution.

13. Two smooth spherical masses m and m' moving with given velocities u and u' in the same direction collide. Shew that the loss of kinetic energy due to the collision is

$$\frac{mm'}{2(m+m')}(u-u')^2(1-e^2),$$

where e is the coefficient of restitution.

*CHAPTER XI.

MOTION IN A CIRCLE. MISCELLANEOUS.

140. The Hodograph. The velocity of a moving particle may be represented at any moment by a straight line drawn from some fixed point, the length of the line represents the magnitude while its direction represents the direction of the velocity. Thus if the particle move with constant velocity the straight line is fixed in magnitude and direction. If the particle move with uniform acceleration in a straight line the direction of the line representing the velocity is fixed; its length however increases uniformly with the time. Thus if OQ represent the velocity at any moment, and if OQ_1

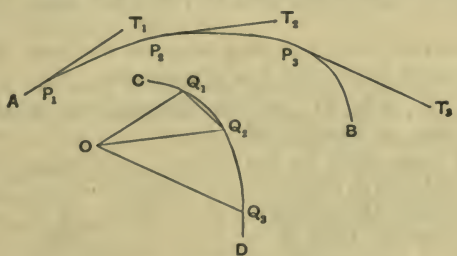


Fig. 90.

represent it after an interval of 1 second, then QQ_1 is the increase of velocity in 1 second, and is therefore equal to the acceleration.

Consider now a particle moving in a curve AB, Fig. 90, let $P_1, P_2 \ldots$ be its positions at different times and draw $P_1T_1, P_2T_2 \ldots$

to represent the velocities V_1, V_2 ... of the particle in those positions. From a fixed point O draw OQ_1 equal and parallel to V_1, OQ_2 equal and parallel to V_2, and so on for the other points P_3 etc.

We thus get a second series of points Q_1, Q_2, Q_3 ... which have the property that the lines drawn to these points from the fixed point O represent the velocities of the particle in the corresponding positions P_1, P_2 etc.

If a similar construction be made for all points on the curve AB we shall get a second curve CD, which has the property that the lines drawn to it from O represent in direction and magnitude the velocity of the particle at the corresponding points of AB.

This second curve CD is called the hodograph of AB.

141. The Hodograph and the Measurement of Acceleration. Again let Q_1, Q_2 in Fig. 90 be two points on the hodograph, P_1, P_2 the corresponding points on the path; in moving from P_1 to P_2 the velocity changes from OQ_1 to OQ_2, join Q_1Q_2; then the change in velocity is represented in direction and magnitude by Q_1Q_2, for Q_1Q_2 represents a velocity which when compounded with OQ_1 will give OQ_2.

PROPOSITION 42. *To shew that in the case of uniform acceleration the hodograph is a straight line.*

If P_1, P_2, Fig. 91, are two positions which the particle occupies after an interval of 1 second, then Q_1Q_2 is the change in velocity in 1 second; if we know that the acceleration is constant then it is measured by the change in velocity in 1 second. Hence in this case the line Q_1Q_2 represents the acceleration. Now let P_3 be the position of the particle after a further interval of 1 second and Q_3 the corresponding point on the hodograph. Then in the same way Q_2Q_3 represents the acceleration, but since the acceleration is

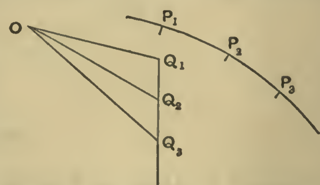


Fig. 91.

constant in direction and magnitude Q_2Q_3 must be equal to and in the same straight line as Q_1Q_2. Thus the hodograph $Q_1Q_2Q_3 \ldots$ must in this case be a straight line.

Hence if a particle move with uniform acceleration the hodograph is a straight line, and the arc of the hodograph traced out in 1 second represents the acceleration.

This proposition can be generalised thus.

PROPOSITION 43. *If* P *be a particle describing any curve and* Q *the point on the hodograph which corresponds to* P, *then the velocity of* Q *in the hodograph measures the acceleration of* P.

For suppose that after a short time τ, P, Fig. 92, has moved to P', and Q in consequence to Q'.

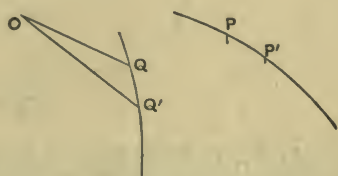


Fig. 92.

Then the velocity of Q is given by the ratio QQ'/τ, when τ is made very small. For QQ' is the space traversed by Q in time τ, and the ratio of the space traversed to the small time of traversing it measures the velocity of Q.

But when τ is very small, we may treat QQ' as a straight line: moreover we may consider the acceleration of P as constant for the interval τ, if that interval be made sufficiently small.

Now OQ is the original velocity of P and OQ' is its velocity after the interval τ. Hence QQ' is the change in velocity during the interval and the ratio of this change to the interval in which it occurs measures the acceleration of P.

Thus the ratio QQ'/τ measures the acceleration of P as well as the velocity of Q. Hence the acceleration of P is equal to the velocity of Q.

Whenever then we know the hodograph of a path and the velocity with which it is described, we can find the acceleration in the original path.

Thus when the hodograph is a straight line described with uniform velocity, the acceleration is constant both in direction and magnitude;

when the hodograph is a straight line but the velocity in it is not uniform the acceleration is constant in direction but variable in amount.

We proceed to give some other examples.

PROPOSITION 44. *A particle describes a circle with uniform speed* V; *to find the hodograph and the acceleration.*

Let the path of the particle be a circle P_1P_2 with centre C, Fig. 93. Let P_1 be the position of the particle in the original path, P_2 its position after one second.

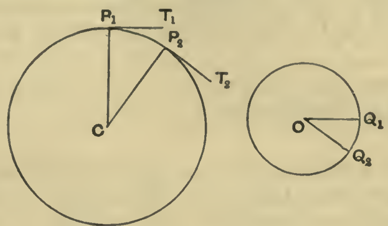


Fig. 93.

At P_1, P_2 draw P_1T_1, P_2T_2 to represent the velocity: since the speed is constant $P_1T_1 = P_2T_2 = V$.

Again, since the path is described with uniform speed and P_1P_2 is the distance traversed in a unit of time we have P_1P_2 equal to V.

Now let OQ_1 equal and parallel to P_1T_1 represent in direction and magnitude the velocity at P_1, and let OQ_2 represent it at P_2.

Then OQ_2 is parallel to P_2T_2. Hence OQ_1 and OQ_2 are respectively perpendicular to CP_1 and CP_2.

Hence the angle Q_1OQ_2 is equal to the angle P_1CP_2.

Since the speed is constant the length of the radius vector from O is constant. Thus the hodograph is a circle and this circle is described by the point Q with uniform speed.

Now Q_1Q_2 represents the space traversed in 1 second by the point in the hodograph when moving with uniform velocity. It is therefore equal to the velocity in the hodograph. Thus it

represents the acceleration completely. Also since the velocity at Q_1 is perpendicular to OQ_1, the acceleration at P_1 is perpendicular to P_1T_1, that is, it is along P_1C. Hence the acceleration at P_1 is represented in amount by Q_1Q_2, the arc of the hodograph described in one second, and is directed along P_1C.

Hence if a be the acceleration we have

$$Q_1Q_2 = a,$$
$$OQ_1 = V.$$

Let r be the radius of the circle P_1P_2.

Then since the angles Q_1OQ_2 and P_1CP_2 are equal their circular measures are the same.

Hence $$\frac{Q_1Q_2}{Q_1O} = \frac{P_1P_2}{P_1C},$$

or $$\frac{a}{V} = \frac{V}{r}.$$

Thus $$a = \frac{V^2}{r}.$$

Hence when a particle moves with uniform speed V in a circle of radius r, it has at each point an acceleration directed to the centre of the circle and equal in amount to $\frac{V^2}{r}$.

Thus the force towards the centre is

$$\frac{mV^2}{r},$$

if m is the mass of the particle.

By expressing these results in terms of the angular velocity of the particle we can put them in slightly different form, for if Ω be the angular velocity we have (Section 38)

$$V = \Omega r.$$

Hence $$a = \frac{V^2}{r} = \Omega^2 r,$$

$$F = \frac{mV^2}{r} = m\Omega^2 r.$$

Thus when we observe a body moving in a circle with uniform speed we know that it has the acceleration given above.

15—2

If a stone is tied to a string and swung round in a horizontal circle, force toward the centre is exerted by the string; this force measures the tension of the string; if we call it T then

$$T=\frac{mV^2}{r}=m\Omega^2 r.$$

The string breaks when the angular velocity is such as to make this tension greater than it can bear.

Example. *A string* 1 *metre long can support a body whose mass is* 10 *kilogrammes. A mass of* 100 *grammes is tied to one end and whirled in a horizontal circle making one revolution per second; find the tension of the string; find also the greatest number of revolutions per second which can be given to the mass without breaking the string, and calculate the kinetic energy of the mass when moving with this greatest possible speed.*

When the mass makes 1 revolution per second, an angle whose circular measure is 2π is traced out in each second; thus the angular velocity is 2π.

Hence since $r=100$ cm. the acceleration is

$$4\pi^2\, 100 \text{ cm. per sec. per sec.},$$

and the force is

$$4\pi^2\, 100\times 100,$$

or

$$394880 \text{ dynes approximately.}$$

When the mass makes n revolutions per second, the angular velocity is $2\pi n$, and the tension

$$4\pi^2 n^2 10^4 \text{ dynes},$$

or approximately

$$394880 n^2 \text{ dynes.}$$

Now the breaking tension is the weight of 10 kilogrammes or

$$10\times 981\times 1000 \text{ dynes.}$$

Hence to find the maximum number of revolutions per second we have

$$394880 n^2 = 981\times 10^4.$$

Thus

$$n^2=\frac{981000}{39488}=24\cdot 85.$$

Hence $n=4\cdot 98$ or very nearly 5.

Hence the string will break before the mass attains a speed of 5 revolutions per second.

The kinetic energy is $\frac{1}{2}mV^2$ ergs.

The value of this is

$$\tfrac{1}{2}\times 4\pi^2\times 100 n^2\times 100^2 \text{ ergs},$$

or

$$490\cdot 8\times 10^6 \text{ ergs.}$$

142. Motion in a circle. Consider a body such as a marble in a horizontal tube along which it can move freely: if the tube be set spinning about a vertical axis, the marble will be shot out at one end; now suppose the marble attached by a spiral spring or piece of elastic to some point in the tube, the spring will be stretched until the tension it can exert on the marble and the impressed force mv^2/r on the marble are equal.

Suppose now that the marble is attached to two points in the tube as shewn at AB, Fig. 94, by two spiral springs, the

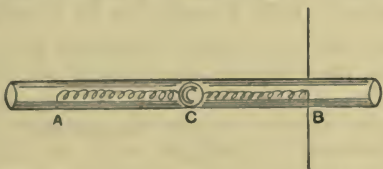


Fig. 94.

marble being between A and B. Both these springs are stretched until each is subject to the same tension. The one spring AC may represent a spiral spring balance by which the marble is suspended, the tension in the other spring will then stand for the force of attraction between the earth and the marble, the weight of the marble; this weight is measured by the extension of the balance AC. Now set the tube rotating about a vertical axis through B, the marble will move towards A, the spring AC will be less stretched than before, the weight of the marble as measured by the extension of this spring will appear less. If when the tube were at rest the spring AC were cut, the marble would move towards B with an acceleration depending on the extension of the spring BC; if when the tube is rotating the spring AC be cut, the marble unless the rotation be too great will move towards B, but its acceleration will be less than it was before by the amount $\Omega^2 r$, where Ω is the angular velocity and r the distance from the axis of rotation.

The acceleration with which the marble moves towards B stands for the acceleration with which a body falls under

gravity; this, other things being equal, is greater when the tube is at rest than it is when it is moving round an axis.

143. Consequences of the Earth's Rotation. Now let us apply this to the Earth. A particle on the Earth is describing a circle about the axis of the Earth; near the pole this circle is small, near the equator it is large, the angular velocity however is the same in all cases. We may illustrate this in a rough manner by supposing that in order to represent a particle near the pole with the tube, the axis of rotation passes near to the marble, while to represent a particle near the equator the axis of rotation is far from the marble.

In the first case the marble will be very slightly disturbed by the rotation, its weight as measured by the spring AC will be nearly the same as it was when the tube was at rest; in the second case the marble may be considerably displaced, its weight is appreciably diminished; thus we see how in consequence of the rotation of the Earth the acceleration of a falling body is less near the equator than near the pole.

144. Circular Motion. It follows then that in order that a body may move in a circle with uniform speed, its connexion with some other body must be such as to be consistent with its having an acceleration v^2/r towards the centre of the circle, if this is not the case the body will not move in the circle.

When the marble is free in the tube, it cannot, under the normal pressure of the walls, acquire an acceleration towards the centre and is shot out of the rotating tube; when it is connected to the spring, the spring is stretched until this acceleration is acquired and then the circular motion continues.

Many other examples might be given. Thus consider a body C suspended by a vertical rod BC from a horizontal arm AB: let there be a joint at B which will allow the body to move in the vertical plane ABC. Cause the whole to rotate about a vertical axis through A, the body C will rise, the rod BC no longer remaining vertical. For if the rod be vertical the body is in equilibrium under its weight and the tension of the rod, it cannot then have an acceleration towards the centre of

rotation on the axis through A. When the body rises the tension is no longer vertical and we can resolve it into two

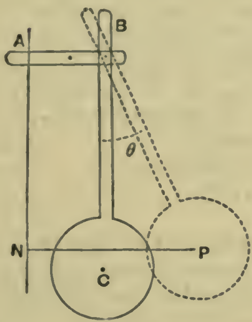


Fig. 95.

components, the one horizontal, the other vertical: since when it is rotating steadily the body has no vertical acceleration, the vertical component of the tension balances the weight: under the horizontal component the body acquires the acceleration $\Omega^2 r$ necessary for its circular motion.

We can find the position of the body thus: let P be its position and let θ be the angle which the rod PB makes with the vertical. Let $AB=a$, and $BP=b$. Draw PN on the vertical through A. Let m be the mass of the body, T the tension of the rod.

Then $$r=PN=a+b\sin\theta.$$

Hence resolving horizontally

$$T\sin\theta=m\Omega^2 r=m\Omega^2(a+b\sin\theta),$$

$$T\cos\theta=mg.$$

Therefore

$$\tan\theta=\frac{(a+b\sin\theta)\,\Omega^2}{g}.$$

From this equation we can find θ, and then θ being known the tension is given by the equation

$$T=mg\sec\theta.$$

An arrangement of this description is made use of in the ball governors attached to some forms of steam-engine. Mechanism for opening and closing the steam ports so as to vary the supply of steam is connected with the ball. When the engine runs too fast the ball rises and the steam port is closed, when the speed is too slow the port is opened, for the ball falls. Watt's governor is shewn in Fig. 96. In this form there are two balls, and the value of a is zero.

Hence $\quad T = \Omega^2 mb = mg \sec\theta.$

Therefore for the equilibrium position

$$\cos\theta = g/\Omega^2 b.$$

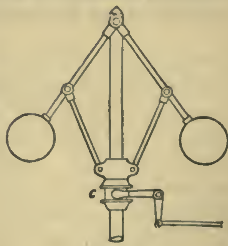


Fig. 96.

145. The shape of the Earth. When dealing with the variation of g attention was called to the fact that the Earth was not spherical but was flattened at the poles; this again is a consequence of the rotation. This may be shewn by spinning, about a vertical axis, a circular hoop, such as that shewn in Fig. 97, of thin brass or some other elastic material.

The hoop is fixed at the bottom to the axis, at the top the axis passes through a collar attached to the hoop in which it can slip freely.

When the hoop is rotating with uniform speed there must be a force on each particle of mass m at a distance r from the axis equal to $m\omega^2 r$.

Fig. 97.

Unless this force is exerted the particle cannot move uniformly in a circle. When the motion is first started the action between the various parts of the hoop is not such as to give rise to this force. At first therefore each particle does not move uniformly in a circle, it also moves outwards from the axis; by this motion the hoop is bent from its circular form, becoming flattened at the top and bottom, and this bending continues until the acceleration acquired by the particles under the forces to which the bending gives rise is that requisite to give uniform motion in a circle.

The same may be shewn by floating a spherical bubble of

oil in a mixture of alcohol and water. Such a bubble can be set into rotation, and when this is done it becomes flattened at the points in which its surface is met by the axis. Now the material of which the Earth is composed resembles in some respects the brass hoop or the oil of the bubble, the Earth is rotating round its axis and hence its shape is not spherical but oval, flattened at the poles.

146. Simple harmonic motion. Let ACB, Fig. 98, be a diameter of a circle centre C, and suppose a particle P is moving round this circle with uniform angular velocity Ω. Let it start initially from the point A and let it be at the point P after t seconds.

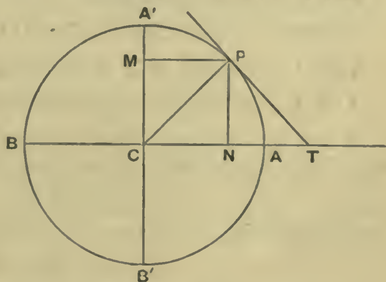


Fig. 98.

Let $A'CB'$ be the diameter perpendicular to ACB and draw PN, PM perpendicular to AB and $A'B'$ respectively.

Then in t seconds the radius CP traces out the angle PCA.

Hence $PCA = \Omega t$.

Let $CP = a$.

Then $CN = CP \cos PCA = a \cos \Omega t$,

$CM = CP \sin PCA = a \sin \Omega t$.

Now as P moves uniformly round from A, the point N starts from A and moves along AC towards C; when P is at A', N has reached C; as P moves on to B, N moves towards B, coinciding with P at B; as P moves back along BB' to A, N moves back to A. Thus N has a vibratory motion backwards and forwards along AB and its distance x from C is always given by the equation $x = a \cos \Omega t$.

The motion of N is said to be simple harmonic *motion.*

The motion of M is also simple harmonic.

PROPOSITION 45. *To find the velocity of a particle moving with simple harmonic motion.*

The velocity of P, Fig. 98, may be resolved into two components, one along AC and the other along CA', the former of these gives the velocity of N.

Now the velocity of P is V at right angles to CP. Draw PT at right angles to CP meeting CA produced in T; then a velocity V along TP has for its component along AC a velocity

$$V \cos PTC.$$

But CPT is a right angle.

Hence $$\cos PTC = \sin PCA = \sin \Omega t.$$

Thus the velocity of N is $V \sin \Omega t$.

Also $V = a\Omega$; we have therefore the result that

When the distance of a particle, oscillating about a fixed point, from that fixed point is given by

$$x = a \cos \Omega t,$$

then the velocity of the particle toward that point is given by

$$v = a\Omega \sin \Omega t.$$

PROPOSITION 46. *To find the acceleration of a particle moving with simple harmonic motion.*

The acceleration of P is $\Omega^2 a$ towards C. This can be resolved into $\Omega^2 a \cos PCA$ along NC and $\Omega^2 a \sin PCA$ along MC. The first of these is the acceleration of N.

Hence the acceleration of N is given by

$$\Omega^2 a \cos PCN \text{ or } \Omega^2 a \cos \Omega t.$$

But $$CN = a \cos PCN.$$

Thus the acceleration required is

$$\Omega^2 CN \text{ or } \Omega^2 x.$$

We have thus the result that when the acceleration of a particle moving in a straight line is always directed to a fixed point in the line and is proportional to the distance of the particle from that point, then the motion is simple harmonic.

Moreover if the acceleration at any distance x is equal to μx, then the value of x in terms of the time is given by

$$x = a \cos \sqrt{\mu}\, t,$$

where a gives the distance of the particle from the fixed point initially.

This follows from the result that if the distance is

$$x = a \cos \Omega t,$$

then the acceleration is $\Omega^2 x$; hence if the acceleration is μx we have

$$\Omega^2 = \mu \text{ and } x = a \cos \sqrt{\mu}\, t.$$

Moreover if T be the time of a complete revolution, then in T seconds the radius traces out four right angles: hence

$$\Omega T = 2\pi.$$

Thus if the acceleration at distance x be μx,

$$\text{the Periodic time} = \frac{2\pi}{\Omega} = \frac{2\pi}{\sqrt{\mu}}.$$

Thus if a particle move in a straight line under an acceleration μx towards a fixed point the motion is a simple harmonic vibration; the distance x of the particle from the fixed point at any time t is given by $x = a \cos \sqrt{\mu}\, t$, its velocity towards the fixed point is $v = a\mu \sin \sqrt{\mu}\, t$, and T the time of a complete oscillation is found from the equation

$$T = \frac{2\pi}{\sqrt{\mu}}.$$

It is not necessary for the motion to take place in a straight line. We may suppose the particle to be a ring oscillating backwards and forwards on a smooth straight wire, the acceleration being directed to a fixed point on the wire. Then the wire may be bent into any shape without altering the motion, provided that the acceleration at each point of the wire remain of the same value as before. This will be secured if the acceleration be proportional to the distance of the particle from the fixed point measured along the wire, the particle still having the same harmonic motion as before. Thus a particle moving on a smooth curve with an acceleration directed along the curve to some fixed point on the curve, and equal to $\mu \times$ {distance measured along the curve of the particle from the fixed point}, has simple harmonic motion about the point, the period of the motion being $2\pi/\sqrt{\mu}$.

Example. *A particle suspended by a string of length* l *oscillates in a vertical circle of radius* l; *find the time of a complete oscillation.*

Let the string CP, Fig. 99, make an angle θ with the vertical line CA.

Let A be the equilibrium position, and let the length of the arc AP be s.

The acceleration of the particle along the curve is $g \sin\theta$.

Now if θ be small $\sin\theta = \theta = s/l$.

Hence acceleration of particle $= \frac{g}{l} s$.

Thus the motion is simple harmonic, and

$$s = s_0 \cos\sqrt{\frac{g}{l}}\, t,$$

while the periodic time is $2\pi\sqrt{\frac{l}{g}}$.

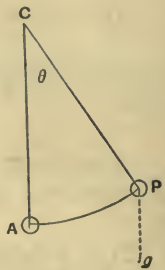


Fig. 99.

This example gives us the case of a simple pendulum, and we have thus proved the formula for the time of swing which was verified by experiment in Experiment 29. We see moreover that this formula is only approximate, for the motion is not simple harmonic unless θ is so small that we may treat θ and $\sin\theta$ as equal. For other applications of this method of dealing with problems in simple harmonic motion, see Maxwell, *Matter and Motion*, Article CXVI. and following.

EXAMPLES.

CIRCULAR AND HARMONIC MOTION. THE PENDULUM.

1. Explain what is meant by simple harmonic motion.

Describe some method of causing a particle to move with simple harmonic motion, and determine the energy of such a particle.

2. A mass of one pound is attached to a string three feet long and whirled in a horizontal circle. If the string can just carry a weight of ten pounds, find how many revolutions per second the mass can make without breaking the string.

3. A clock whose pendulum ought to beat seconds gains at the rate of 10 minutes a week. What alteration must be made in the length of the pendulum to correct its error? The length of the simple seconds pendulum is about 39·12 inches.

4. Define the terms Acceleration, Momentum, Kinetic Energy, and determine their values in the case of a simple pendulum which has been drawn aside through a known angle and then released.

5. A pendulum consisting of a bob weighing 1 kilogramme at the end of a string 1 metre long is drawn aside until the bob is 25 cm. from the vertical through the point of support, and is held in this position by a horizontal string. Find the forces on the bob (1) when in this position, (2) just after the horizontal string is cut, (3) as the bob swings through its lowest position.

6. Shew that there is a force acting on a body which moves with uniform velocity in a circle.

7. A pendulum which beats seconds at a place A gains 10 seconds a day when taken to another place B; compare the intensity of gravity at A and B.

8. Describe an arrangement to exhibit the combination of two simple harmonic vibrations in planes at right angles. Point out how such an arrangement may be used to illustrate or verify the law of the addition of vector quantities.

9. Define a simple harmonic vibration and shew that such a vibration is executed when a particle moves under the action of a force which varies as the distance of the particle from its position of equilibrium.

10. A bullet fired from a gun hits the bob of a heavy simple pendulum and remains imbedded in it. The masses of the bullet and of the pendulum bob are known, shew how by observing the amplitude of the first swing of the pendulum the velocity of the bullet may be found.

11. A particle is describing simple harmonic oscillations in a straight line, shew how to determine graphically its velocity, and prove that the potential energy of the particle is proportional to the square of its distance from its equilibrium position.

12. Describe the motion of the bob of a pendulum if the upper part of the string is doubled and the two ends are attached to two separate points in the same horizontal line.

Under what circumstances does the bob describe a path analogous to that of a falling stone?

13. What are isochronous vibrations?

Shew that a body will execute isochronous vibrations if its potential energy varies as the square of its distance from its position of equilibrium.

14. The mass of a simple pendulum of length 3 ft. is 2 lb. The pendulum is raised till it makes an angle of 30° with the vertical and let go; find its energy.

15. Two equal masses are attached to the ends of a string passing through a hole in a smooth horizontal table. One mass moves uniformly on the table in a circle round the hole at its centre, while the other mass is thus kept hanging at a constant distance below the hole. Find the velocity of the moving mass if the radius of the circle be 6 yards.

16. A mass of 1 gramme moves with a harmonic motion and vibrates 128 times a second through a range of 1 cm. Find the energy it possesses.

17. A mass of m pounds is suspended from a point by a string of length l feet. The mass revolves in a horizontal circle with the string inclined to the vertical and makes n revolutions per second. Shew that the string is inclined to the vertical at an angle whose cosine is

$$\frac{g}{4\pi^2 n^2 l}.$$

MISCELLANEOUS EXAMPLES.

1. What is meant by 1 kilogramme and by 1 oz.? Describe carefully the observations you would make to determine the number of ounces in 1 kilogramme if given a balance, not known to be in adjustment, and a set of weights.

2. What is the "Standard yard"?

Describe carefully the method you would employ to determine the length of a given yard measure in terms of the Standard, and to check the accuracy of its inch divisions.

3. What is the unit of force in absolute measure? If the unit of mass be changed from a gramme to a kilogramme and the unit of time from a second to an hour, in what proportion is the measure of the weight of a given body affected?

4. What do you understand by an absolute system of units and a derived unit?

What would be the unit of time of the British absolute system if the weight of one pound at London were to be the absolute unit of force, the units of length and mass remaining unaltered?

What would be the advantages and disadvantages of such a change?

5. What is the unit of force on the C.G.S. system? and what is the unit of work? What are the respective units commonly employed in England?

6. If a nation uses 39 inches as unit length, 3 seconds as unit time, and 1 cwt. as unit mass, what is the unit force in lb. weight?

7. The measure of a certain power is 10 when 1 ft., 1 sec., and 1 lb. are the units. What is its measure when 1 yd., 1 minute, and 1 ton are the units?

8. Define velocity. What do we mean when we say that a train is travelling at the rate of 60 miles an hour?

Find the direction of the blow which drives a cricket-ball to square-leg with the same velocity as that with which it reaches the bat.

9. A stone is dropped into a well from the surface of the ground. The sound of its reaching the water is heard $5\frac{3}{4}$ seconds after the stone was dropped. If the velocity of sound is reckoned as 1000 feet per second, find the depth of the well. ($g=32$ feet per second in each second.)

10. A traveller alights from a tramcar, which is traversing a straight street, and starts to walk at 4 miles an hour along a straight side-street; after walking 10 minutes he reaches a street crossing his own at right angles, and, looking along it, he sees his tramcar at the end of it, half-a-mile away. Find the velocity of the tramcar; and draw a diagram shewing the inclination of the streets.

11. Two trains are moving on two lines inclined at a small angle to one another, with the same velocity. Shew that if an observer in one train fix his eye upon a particular point of the other train, this other train may seem to be moving faster or slower than his own, or at the same rate, according to the direction, relative to him, of the point of the other train which he is watching.

12. The horizontal velocity of a shot is 1100 feet per second and the range 3000 yards: find the initial velocity.

13. Two ships are sailing in directions making an angle of 60° with each other, with velocities of 15 and 20 miles an hour respectively. Find the magnitude of the velocity of one ship relative to the other.

14. How is the measure of an acceleration altered when the unit of time is changed from a second to a minute?

15. A train uniformly retarded has its velocity reduced from 30 to 24 miles per hour in 15 minutes. Find how far it goes in the interval.

16. Find the acceleration of a train, supposing it uniform, which passes one station at the rate of 20 miles an hour, and another 5 miles distant at the rate of 30 miles an hour.

17. A stone is thrown from a railway train with such a velocity in a direction at right angles to the path of the train that, relatively to the ground, it has a velocity of 30 miles an hour in a direction making 30° with the path of the train. What is the velocity of the train?

18. A particle moving with uniform acceleration has at a given instant a velocity of 33 feet per second; eleven seconds later it has a velocity of a mile per minute. Determine the measure of the acceleration, a foot and a second being the units of length and time.

19. With a foot and a second as units of length and time, the measure of an acceleration is 49; find the unit of time when the measure of the same acceleration is 12, and the unit of length a yard.

20. The line AB is vertical, and ACB is a right angle. Shew that the time of sliding down either AC or CB, supposed smooth, is equal to the time of falling down AB.

21. A train is moving at a rate of 60 miles an hour, and a gun is to be fired from a carriage window to hit an object which at that moment is exactly opposite the window. If the velocity of the bullet be 440 feet per second, find the direction in which the gun must be pointed.

22. A bag of ballast is dropped from a balloon when the balloon is ascending at the rate of 10 feet per second and is at a height of 300 feet; find the time occupied in the bag's descent.

(The acceleration due to gravity measured in feet and secs. may be taken as 32.)

23. A boy throws a stone 150 feet vertically upwards. What was the velocity of the stone when it left the boy's hand?

24. Compare the velocities of two trains, one of which is moving at the rate of 66 feet per second, and the other at the rate of 40 miles an hour.

25. How is uniform acceleration measured? If the measure of the acceleration due to gravity be 32, the foot and the second being the units of length and time; find its measure when 2 feet and $\frac{1}{4}$ second are the units of length and time.

26. A boat is set with her head due N.E. Under the action of the wind alone the boat would move in a N.E. direction with a velocity of $4\sqrt{2}$ miles per hour. The tide is flowing due south at a rate of 4 miles per hour. Shew that the boat's actual course is due east.

27. A cricket-ball is thrown vertically upwards with a velocity of 56 feet per second. Find the velocity when it is half-way up, and the height to which it has risen when half the time to the highest point has elapsed. (The resistance of the air is neglected, and the acceleration of gravity = 32 feet per second in each second.)

28. A man starts at right angles to the bank of a river, at the uniform rate of $1\frac{1}{4}$ miles per hour, to swim across; the current for part of the way is flowing uniformly at the rate of 1 mile per hour, and for the remainder of the way at double that rate. He finds when he reaches the other side that he has drifted down the stream a distance equal to the breadth of the river. At what point did the speed of the current change?

29. Shew that the difference of the square of the velocities at any two points, of a body falling in vacuo, varies as the distance between them.

A body falls from rest in vacuo through a certain height and acquires a certain velocity. Find how much further the body will have fallen when it has doubled its velocity.

30. A stone is thrown vertically upwards with a velocity of 36 feet per second. To what height will it rise, and after what intervals of time will it have a velocity of 12 feet per second?

31. A stone is dropped from a height of 8 feet above the ground from the window of a railway carriage travelling at the rate of 15 miles an hour; find its velocity on striking the ground.

32. If a small smooth ball be set rolling up an inclined plane in a direction other than the line of slope, find the curve which it will describe.

33. A projectile weighing half a ton is fired with a velocity of half a mile a second from a 100-ton gun. Find the velocity of recoil of the gun, and compare its kinetic energy with that of the projectile.

34. Two masses of 3 lb. and 4 lb. respectively are connected by a string passing over a pulley. Find the acceleration and tension of the string.

35. A 50-ton engine moving at the rate of 10 miles an hour impinges on a truck at rest weighing 10 tons, and the two move on together. Find their velocity and calculate the loss of kinetic energy.

36. Two equal weights of 1 lb. are connected by a fine string passing over a light pulley. A weight of 1 oz. is attached to one of them. Find the acceleration and the tension of the string.

37. A ball 1 lb. in mass, with coefficient of restitution $\frac{3}{4}$, is let fall to the ground from a height of 32 feet. Find the loss of kinetic energy on impact, and the height to which the ball will rebound.

38. In the system of pulleys in which each pulley hangs by a separate string and all the pulleys are of the same weight, find the acceleration of the weight when there are n pulleys, all the strings being vertical.

39. A fly-wheel is brought to rest after n revolutions by a constant frictional force applied tangentially to its circumference. If k be the kinetic energy of the wheel before the friction is applied and r its radius, shew that the friction is $k/2\pi nr$.

40. A lump of clay weighing 10 lb. is thrown with a velocity of 50 feet per second against an equal lump at rest: if the two travel together with a velocity of 25 feet per second, find the loss in energy estimated in foot-pounds.

41. Find the amount of work done in drawing a weight of 3 tons 100 yards along a rough horizontal plane, when the friction is 25 lb.-wt. per ton.

42. A colliery engine draws 10 tons of coal up a shaft 1000 feet deep in 1 minute. Find the total amount of work done and average power given out by the engine.

43. The handle of a hoisting-crab moves through 1 foot while the weight lifted moves through $\frac{1}{4}$ inch. It is found that to raise 1 ton a force of 80 lb.-wt. must be applied to the handle. What proportion of the work is spent in overcoming friction?

44. How much work is done in elevating 2 cwt. of coals from the bottom to the top of a staircase, the staircase having 66 steps and each step 8 inches high?

45. Two stones are thrown at the same instant from the tops of two towers directly at one another; shew that, neglecting the effect of the atmosphere, they will meet if the velocities of projection are great enough.

46. A particle is moving in a circle with constant speed. Assuming the mass and speed of the particle and the radius of the circle to be known, state the force acting on the particle and its acceleration.

47. A shot 8 lb. in mass leaves a gun 18 tons in mass with a velocity of 1500 feet per second; with what velocity does the gun recoil?

48. A constant horizontal force will give to a mass of 15 lb., starting from rest and supported on a smooth horizontal plane, a velocity of 28 feet per second in 4 seconds. What weight will that force support?

49. A force equal to the weight of 10 lb. acts for a minute on a mass of 1 cwt. Find the momentum and energy of the mass. What is the work done by the force?

50. A truck which weighs 10 tons is free to move without friction in a horizontal direction under the action of a horizontal force equal to the weight of 42 lb.; find the acceleration and determine how fast (in miles per hour) the truck would be moving if the force continued to act for an hour.

51. How fast must the bob of a pendulum (length 3 ft.) be moving at the bottom of its swing if it is to reach the horizontal through the point of support before it turns?

Will the required velocity be greater for a small bob than for a large one? Give reasons for your answer.

52. Determine the kinetic energy lost in the direct impact of two elastic spheres, each of a pound mass, moving in opposite directions, each with a velocity of 1 mile in 3 minutes; the coefficient of restitution being $\frac{1}{2}$.

53. An engine of 45 horse-power is drawing a train; if the resistance to the motion when moving with a velocity of v feet per second be $\frac{1}{25}v^2$ lb. weight, find the maximum velocity the train can attain. [1 horse-power = 550 ft.-lb. per second.]

54. Find the number of foot-pounds of work which must be done on a fly-wheel whose mass is 1000 lb. and radius 30 inches to give it a velocity of 600 revolutions per minute, assuming the whole mass of the wheel to be concentrated in the rim.

55. Shew that if a body falls freely under gravity there is neither loss nor gain of energy; and explain how to apply the same principle to find the velocity of a body sliding down a smooth curve.

EXAMINATION QUESTIONS.

I.

1. What are the units in terms of which length, mass and time are usually measured (1) in England, (2) on the Metric System?

2. Define the terms Motion, Velocity, Speed, Acceleration, and explain how they are measured.

3. What is meant by the composition of Velocities? Enunciate and prove the parallelogram of Velocities.

4. Shew that in the case of a particle moving with uniform acceleration a in a straight line the following relations hold

$$v = at \qquad s = \tfrac{1}{2}at^2 \qquad v^2 = 2as,$$

v being the velocity at the end of the time t, and s the space passed over.

5. State Newton's Laws of Motion, explaining the terms used in your statement.

6. Define Force and shew how the second law enables us to obtain a measure of force; prove the formula $F = ma$, where a is the acceleration produced in mass m by the force F. In what units must these various quantities be measured if the above formula is to hold?

II.

1. State the second law of motion and shew clearly how to use it to determine the measure of the unit of force. Define the terms dyne, poundal.

2. What experiments are required to shew that the weight of a body is proportional to its mass?

3. Describe Atwood's machine and shew how to use it to verify the formulae $s = vt$, $v = at$, $s = \frac{1}{2}at^2$, $F = ma$, with the usual notation.

4. How would you verify the fact that the time of fall of a body is independent of its horizontal motion and that the path of a projectile is a parabola?

5. What is meant by a simple pendulum? Shew that the time of swing of such a pendulum is proportional to the square root of its length. What inference do you draw from the fact that it does not depend on the mass of the bob?

6. State the third law of motion.

Define the terms work and energy; and shew that if a body of mass m has a velocity v produced in it by the action of a constant force F, then the work done by the force is $\frac{1}{2}mv^2$.

7. Shew that if a body is falling freely under gravity its energy remains constant.

III.

1. Explain the terms Work and Energy and shew how work is measured; distinguish between an erg, a foot-pound and a foot-poundal.

What do you understand by Power? What is a Horse-power?

2. Distinguish between kinetic and potential energy and shew how they are measured in the case of a falling body.

Shew that if a body be let fall from rest its energy remains constant till it reaches the ground.

3. State and prove the parallelogram of forces.

4. What is meant by the Resolution of forces? Shew how to find the resolved part of a force in two directions at right angles.

5. A body is projected in a horizontal direction; shew that its path is a parabola and find its latus rectum.

6. Describe experiments to determine the relation between the relative velocities before and after impact of two balls which impinge directly.

7. Shew that a body moving with uniform speed v in a circle of radius r has acceleration equal to v^2/r.

8. Explain what is meant by simple harmonic motion.

STATICS

STATICS.

CHAPTER I.

FORCES ACTING AT A POINT.

1. Equilibrium. In **Statics** we consider the relations which must exist between a set of forces impressed on a body if the body is to remain at rest.

DEFINITION OF FORCE. **Force** *is action exercised on a body so as to change or tend to change its state of rest or of uniform motion in a straight line.*

Each force is measured by the acceleration the body would have if the other forces were not impressed on it. The actual acceleration is compounded of these several accelerations which coexist simultaneously; to compound two or more accelerations we use the parallelogram law[1]. The resultant acceleration will be a measure of the resultant force; this is found from the individual forces by the same process as that by which the resultant acceleration was obtained from its components.

2. Representation of a Force. A force can conveniently be represented by a straight line; one end of the line will represent the point of application of the force, the direction of the line gives the direction, or line of action of the force, while the number of units of length in the line measures the number of units of force in the force.

3. Measurement of a Force. A force in Statics may be measured in terms of any convenient unit, the weight of 1 lb. or the weight of 1 gramme are often taken as units, and we speak of a force of P grammes weight or P lb. weight: we use gravitation units.

[1] Dynamics, Sections 30, 96.

If we have to consider the motion due to a force so measured we must remember that the weight of a gramme contains g (981) dynes, where g denotes the acceleration of a falling body in centimetres per second per second, while the weight of a pound contains g (32·2) poundals, g being in this case measured in feet per second per second.

4. Resultant Force. We proceed now to find the resultant of two or more forces impressed on a particle.

DEFINITION. *If two or more forces P, Q... be impressed on a rigid body, the actual acceleration of the body is found by compounding the accelerations communicated by each force separately. If a single force R can be found which, when impressed alone, will communicate to the body this acceleration, this force is called the Resultant of the two or more forces, and these forces are called its components.*

PROPOSITION 1. *When two or more forces are impressed on a particle in the same direction their resultant is the sum (the difference if the forces act in opposite directions) of the forces.*

For let OA, Fig. 1, represent the one force P and AB the second Q, then OA and AB represent also the accelerations of a unit mass on which these forces are impressed. The resultant acceleration is (Dyn. § 29) represented by OB. Thus the resultant force R is represented by OB, and since

Fig. 1.

$$OB = OA + AB,$$
$$R = P + Q.$$

Similarly if the forces be impressed in opposite directions, we have, Fig. 2, $OB = OA - AB$.

Thus $R = P - Q$.

Fig. 2.

If the lines of action of the forces P and Q are not in the same straight line we find the resultant by the parallelogram law.

5. Parallelogram of Forces.

PROPOSITION 2. *If two forces represented in direction and magnitude by two straight lines OA, OB be impressed on a*

particle their resultant is represented by OC the diagonal through O of the parallelogram which has OA and OB for adjacent sides.

Let OA, OB, Fig. 3, represent two forces P and Q impressed on a particle. Complete the parallelogram $AOBC$ and draw the diagonal OC. Then OC shall represent the resultant force R.

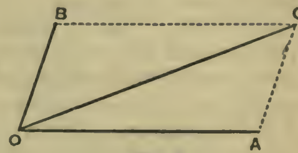


Fig. 3.

For OA, OB represent also the accelerations of a particle of unit mass on which the forces P, Q are each separately impressed; and by the parallelogram of accelerations OC represents the resultant acceleration of such a particle; but the resultant acceleration measures the resultant impressed force.

Hence OC the diagonal of the parallelogram represents R the resultant impressed force.

Thus forces are compounded and resolved according to the parallelogram law and Propositions 9 to 13 of the Kinematics, relating to displacements, apply equally to forces.

We may notice that the resultant force does not depend on the mass of the particle. If in Fig. 3 above, the mass of the particle be m, and the accelerations corresponding to P, Q, and R be p, q, r; then since P is equal to mp, Q to mq, and R to mr, the lines OA, OB represent mp and mq respectively. Thus OC represents mr on the same scale, hence it represents the resultant force R.

The proof of the parallelogram of forces depends therefore on that of the parallelogram of accelerations.

6. Experiments on the Parallelogram of Forces.

The parallelogram of forces can be verified in various ways by direct experiment; we shall describe two such experiments. A student who has difficulty in following the dynamical proof may base his acceptance of the proposition on the direct results of the experiments. A statical proof by Duchayla is often given; there are many reasons however why its use should be avoided and we shall not include it.

In many statical experiments the impressed force is measured by the weight of a body suspended by a string from some part of the apparatus, and it is often necessary to vary this weight. A convenient arrangement is shewn in Fig. 4. An iron rod about 15 cm. in length has a hook at the upper end; to the lower end a flat circular disc is rivetted and the whole is adjusted to have some definite mass such as 1 lb. or, if C.G.S. units are being employed, $\frac{1}{2}$ a kilogramme. The weights take the form of flat circular discs of iron or brass; each disc has a slot cut out as shewn in the figure, the slot reaches to just beyond the centre and is wide enough to admit the vertical rod which supports the scale-pan. Two sizes of "weights," say pounds and $\frac{1}{2}$ pounds or $\frac{1}{2}$ and $\frac{1}{4}$ kilos., will be found convenient; when in use they rest one on the top of the other on the scale-pan forming a pile through the centre of which the supporting rod runs; thus a force equal to the weight of a definite number of half-pounds is easily applied.

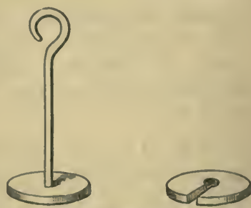


Fig. 4.

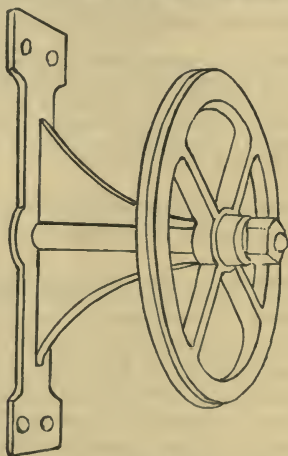


Fig. 5.

In other experiments the force is most easily applied and measured by means of a spring balance.

A useful form of pulley is illustrated in Fig. 5.

EXPERIMENT 1. *To verify by experiment the parallelogram of forces.*

(*a*) The apparatus required for this is illustrated in Fig. 6.[1]

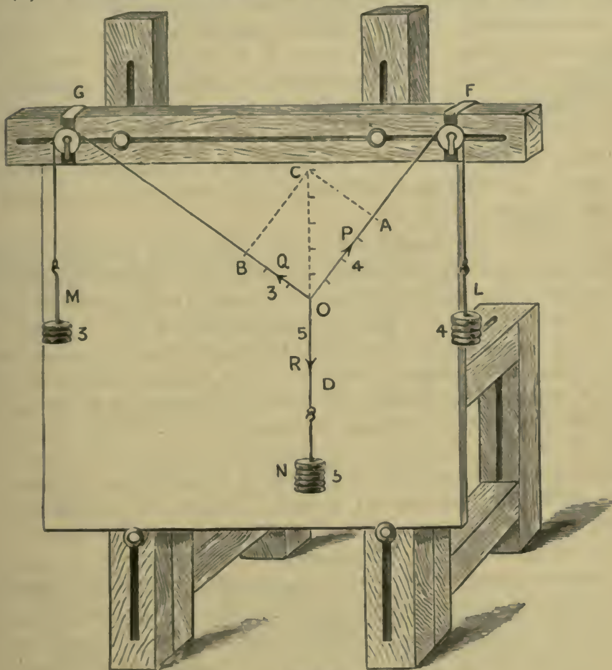


Fig. 6.

Two pulleys *F*, *G* are attached to a horizontal support. A

[1] In the figure the apparatus is shewn supported by a Willis framework. Such a framework consists of a number of bars which can be secured together in various positions by suitable screw bolts, and is very useful for various statical experiments. In a laboratory where a large number of sets of apparatus for the same experiment are required, it is

string AOB passes over these and carries two of the scale-pans just described: these are shewn at L and M. A second string OD knotted at O to the first carries a third pan N.

Some convenient number of weights is put on each scale-pan and the whole system is allowed to come into equilibrium.

Let us suppose the total weights supported at L, M, N respectively including the weight of the scale-pans to be P, Q and R pounds-weight, these weights measure the tensions of the respective strings. Thus forces of P, Q and R lb.-weight act along OA, OB and OD respectively. Since there is equilibrium R is clearly equal and opposite to the resultant of the other two forces P and Q.

Now adopt some convenient length to represent the unit of force, e.g. represent a force of 1 lb.-weight by a length of 10 centimetres.

Draw on the board lines OA, OB, OD, parallel to the strings and measure off along OA and OB lengths OA and OB to represent the forces P and Q; complete the parallelogram $AOBC$ and join OC. Measure the length of OC. It will be found to represent in magnitude the force R on the same scale as OA and OB represent P and Q. Place a straight-edge against OC, it will be found that CO when prolonged is in the same straight line as OD the line of action of R. Thus OC represents R in magnitude but is opposite to it in direction, and since the three forces P, Q, R are in equilibrium, R must be opposite to the resultant of P and Q; hence OC represents the resultant of P and Q represented by OA and OB respectively, and OC is the diagonal of the parallelogram of which OA and OB are adjacent sides. Thus the parallelogram of forces is verified.

By taking along OD a length OD to represent the force R, and constructing a parallelogram with OA and OD as sides, we could shew that Q is represented by the diagonal of this parallelogram.

convenient to have a board fastened to the walls, but projecting some little distance from them, and running round the room at some suitable height. Apparatus such as pulleys, etc. can be secured to this and the weights conveniently suspended from them without coming in contact with the walls. Arrangements should be made for supporting a drawing-board behind the strings which carry the weights, in order to solve questions easily by graphical construction.

In the figure as drawn P, Q, R are forces of 3, 4 and 5 lb. weight. It will be noticed that in this case the angle AOB is a right angle and that since

$$5^2 = 3^2 + 4^2,$$

we have $$R^2 = P^2 + Q^2.$$

(*b*) Knot three strings together at O, Fig. 7, and attach

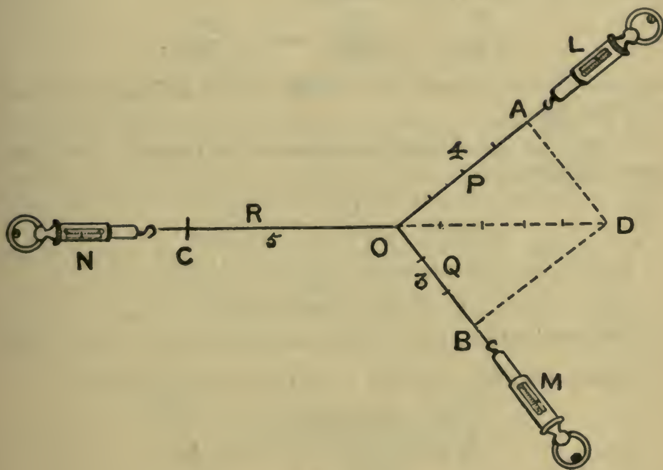


Fig. 7.

their ends to three spring balances L, M, N. Fix the balances to hooks on the edges of a drawing-board in such a way that the strings may be all drawn tight and the balances stretched. The readings of the balances will give us the forces acting along the strings, let them be P, Q, R respectively. Mark off along OL, OM and ON lengths OA, OB, OC to represent the forces P, Q, R; thus if the balances read in lb. we might take a length of 1 inch to represent 1 lb. weight, a force of P lb. weight is then represented by a line P inches in length. Draw lines OA, OB, OC on the paper under the string to represent these forces; then by constructing a parallelogram as before with OA

and OB as sides, we can shew that OC is equal and opposite to the diagonal of this parallelogram, the diagonal represents the resultant of P and Q.

We may use this construction to verify another important formula. Measure with a protractor the angles BOC, COA and AOB opposite respectively to the forces P, Q and R. Look out in a trigonometrical table the sines of these angles and then calculate the values of the fractions

$$\frac{P}{\sin BOC}, \ \frac{Q}{\sin COA} \text{ and } \frac{R}{\sin AOB},$$

the ratio that is of each force to the sine of the angle between the other two.

It will be found that the ratios are all equal. We thus have the equations

$$\frac{P}{\sin BOC} = \frac{Q}{\sin COA} = \frac{R}{\sin AOB}.$$

Thus in the case shewn in Figure 7 in which

$$P = 3, \ Q = 4 \text{ and } R = 5,$$

we find $BOC = 143°$, $COA = 127°$ and $AOB = 90°$.

Hence $\sin BOC = \sin 143° = \sin (180° - 143°) = \sin 37°$

$$= \cdot 6 \text{ approximately.}$$

Also $\sin COA = \cdot 8$; $\sin AOB = 1$.

Hence
$$\frac{P}{\sin BOC} = \frac{3}{\cdot 6} = 5,$$
$$\frac{Q}{\sin COA} = \frac{4}{\cdot 8} = 5,$$
$$\frac{R}{\sin AOB} = \frac{5}{1} = 5.$$

Thus the relation is verified.

7. Further experiments on the equilibrium of forces.

The spring balance may be employed in various other experiments to measure force and verify the results of theory.

Thus in Fig. 8 a weight is supported by two strings each

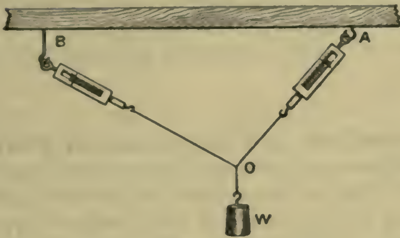


Fig. 8.

of which is attached to a spring balance. The balances are fastened to two points A and B, thus their readings give the tensions in the strings; if a parallelogram be constructed with its sides representing these tensions the diagonal will be vertical and will represent the weight. Moreover we should find in this case that each of the tensions is greater than the weight.

In a similar manner we may support a weight as in Fig. 9 by three or more strings, each of which is attached to a spring balance: the readings of the balances give the tensions, and if,

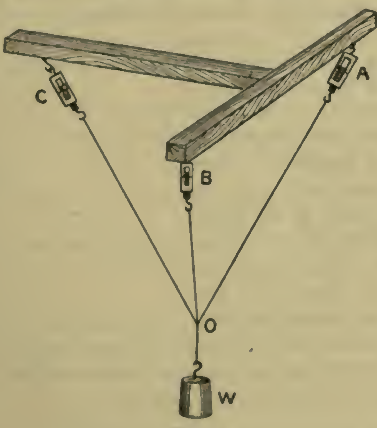


Fig. 9.

starting from any point, we construct a polygon whose sides represent the tensions in direction and magnitude, it will be found that the line joining the starting point to the extremity of the last line so drawn is vertical and represents the weight supported.

8. Composition and Resolution of Forces. Since forces like displacements are combined according to the parallelogram law, the various propositions which have been given for the composition and resolution of displacements apply to forces; for the sake of completeness in this part of the subject we repeat them here. We will first put the parallelogram of forces into a slightly different form.

In order to find the resultant of two forces P, Q acting at a point we draw OA, OB, as in Fig. 3 above, to represent the forces and complete the parallelogram $AOBC$. We have seen that OC is the resultant. Now the position of C can be found somewhat more simply: in the figure AC is equal and parallel to OB, hence AC will represent the force Q in magnitude and direction though not in point of application, for both forces P and Q act at O. We may then clearly find the point corresponding in any given case to C thus.

From O, Fig. 10, the point of action of the forces draw OA to represent the force P, from A the extremity of this line draw AC to represent the force Q in magnitude and direction. Join OC, then OC represents the resultant of P and Q in point of action, magnitude and direction. For by completing the parallelogram by drawing a line from O equal and parallel to AC and another line through C parallel to AO, it is clear that OC is the diagonal of a parallelogram whose two sides meeting at O represent the forces; hence OC represents the resultant of P and Q.

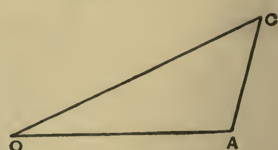


Fig. 10.

This construction can be generalized thus.

PROPOSITION 3. *To find by a graphical construction the resultant of a number of forces impressed on a particle.*

Let OA, OA', OA'', etc. Fig. 11, represent the forces P, P', P'', etc. From A draw AB equal and parallel to OA' to represent Q in magnitude and direction. Then OB is the resultant of P and P'. From B draw BC equal and parallel to OA'' to represent P'' in magnitude and direction: then OC is the resultant of forces represented by OB and BC, and OB represents the resultant of P and P'; hence OC represents the resultant of P, P' and P''.

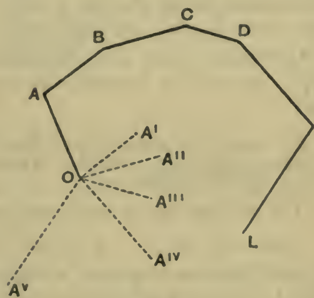


Fig. 11.

Proceeding in this way we find the resultant of any number of forces acting at a point; for if L is the last point found, then the resultant is OL.

Corollary. If L coincide with O the resultant is zero and the forces are in equilibrium. In this case the forces are represented in direction and magnitude by the sides of a closed polygon taken in order and we have the result that:

If a number of forces impressed on a particle be represented in direction and magnitude by the sides of a closed polygon taken in order, the forces are in equilibrium.

This proposition is called the Polygon of forces.

A special case of this is the **Triangle of forces**, of this on account of its importance we give a formal proof.

PROPOSITION 4. *If three forces impressed on a particle be represented in direction and magnitude by the sides of a triangle taken in order, the particle is in equilibrium.*

For let the sides OA, AC, CO, Fig. 12, taken in order, represent in direction and magnitude three forces P, Q, R

acting on a particle at O. By completing the parallelogram $OACB$ we see that OB is equal and parallel to AC and therefore represents Q completely. Hence the resultant of P and Q is represented by OC; this resultant is therefore equal and opposite to R, hence the particle is at rest.

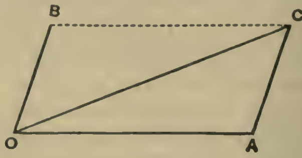


Fig. 12.

It should be noticed that the sides are to be taken in the same direction round the triangle. Thus forces represented by OA, AC, and OC are *not* in equilibrium.

The converse of the above proposition is also true.

PROPOSITION 5. *If three forces impressed on a particle are in equilibrium they can be represented in direction and magnitude by the sides of any triangle drawn so as to have its sides parallel to the forces.*

Let P, Q, R be three forces impressed on a particle at O which are in equilibrium. In Fig. 13 take OA to represent the force P, from A draw AC to represent Q in direction and magnitude and join OC. Then OC represents the resultant of P and Q, and since P, Q and R are in equilibrium, R must be equal and opposite to the resultant of P and Q, thus CO must represent R; hence the forces P, Q, R are represented by OA, AC and CO respectively.

Fig. 13.

Again, any convenient length along OA may be taken to represent P, hence any triangle with its sides parallel to P, Q and R will represent the forces.

The converse of the polygon of forces is not true. All triangles whose sides are parallel to the forces are similar and have their corresponding sides proportional, hence any one of them may be taken to represent the forces: this is not the case for polygons. A number of polygons can be found whose sides represent the forces, all these polygons are similar; but *any* polygon with its sides parallel to the forces will *not* represent them.

The following examples illustrate this graphic method.

Examples. (1) *Find the resultant of forces of* 2 *to the North,* 3 *to the East,* 3 *to the South, and* 4 *to the West, impressed on a particle.*

Draw a vertical line OA (Fig. 14) upwards, 2 cm. in length, to represent the first force. Draw AB to the right, at right angles to OA, 3 cm. in length; BC downwards, at right angles to AB, 3 cm. in length; CD horizontal to the left, 4 cm. in length; then OD is the required resultant.

Also if OL be perpendicular on CD, it is clear that OL is 1 cm. and LD is also 1 cm. Hence OD is $\sqrt{2}$ cm. Thus the resultant force is $\sqrt{2}$ to the South-West. This example is the same as Example 1 on page 43, (Dynamics).

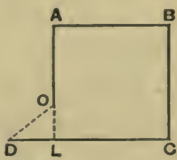


Fig. 14.

(2) *Six forces of* 1, 9, 2, 7, 3, *and* 8 *lb. weight respectively are impressed on a particle in directions parallel to the sides of a regular hexagon taken in order. Shew that the particle is in equilibrium.*

The adjacent sides of a regular hexagon make angles of 120° with each other. Take a line of 1 cm. to represent a force of 1 lb. wt., and draw AB 1 cm. in length to represent the first force, BC inclined at 120° to it 9 cm. in length to represent the second, CD 2 cm. in length to represent the third, and so on. If the figure be carefully drawn, the end of the sixth line representing the force of 3 lb. wt. will be found to coincide with A. The hexagon is a closed one, and the particle is in equilibrium. The student should construct this figure for himself to scale.

Aliter. The forces of 1 and 7 lb. weight acting in opposite directions are equivalent to a force of 6 lb. wt. acting in the same direction as the 7 lb. wt., the forces of 9 and 3 lb. wt. are equivalent to a force of 6 lb. wt. acting in the direction of the 9 lb. wt., the forces of 2 and 8 lb. wt. are equivalent to a force of 6 lb. acting parallel to the 8 lb.; thus we have three equal forces of 6 lb. acting away from the particle in directions inclined to each other at 120°, and these form a system in equilibrium.

(3) *The ends of two strings are secured to two fixed points L and M, and are knotted together at O; a* 5 *kilogramme weight is suspended from O; find by a graphical construction the tensions in the strings.*

Fix a drawing-board in a vertical plane behind the strings, and trace on the board their directions. Take some length (say 5 cm.) to represent a force equal to the weight of 1 kilogramme. From O draw OA (Fig. 15) vertically upwards 25 cm. in length, then OA represents the suspended weight. From A draw AB parallel to OM to meet the string OL in B. Let T_1, T_2 be the tensions in OL and OM; W the weight in direction AO.

Then the three forces T_1, T_2 and 5 kilos weight are parallel to the sides of the triangle OBA taken in order. They are therefore represented by its three sides. Measure the lengths of OB and BA in cm.

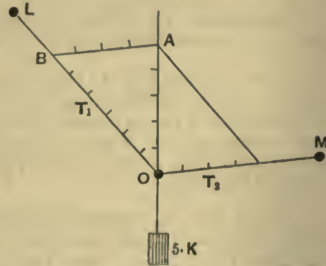


Fig. 15.

Then we have

$$\frac{T_1}{OB}=\frac{T_2}{AB}=\frac{W}{AO}=\frac{5}{25}=\frac{1}{5}.$$

Hence

$$T_1=\frac{OB}{5}\text{ kilos wt., } T_2=\frac{AB}{5}\text{ kilos wt.}$$

In the figure drawn it is clear that

$T_1=6$ kilos wt., $T_2=4$ kilos wt.

(4) *The bob of a pendulum weighing 5 kilos is pulled aside by a horizontal string until its thread is inclined at 30° to the vertical. Find the impressed force in the string and the tension of the pendulum thread.*

Let O (Fig. 16) be the point of suspension, A the pendulum when displaced, OC vertical. Draw AC horizontal meeting OC in C. The impressed forces are 5 kilos weight vertical, parallel therefore to OC, the tension T' of the horizontal string parallel to CA, and the tension T of the pendulum thread parallel to AO. The impressed forces therefore are proportional to the sides of the triangle AOC.

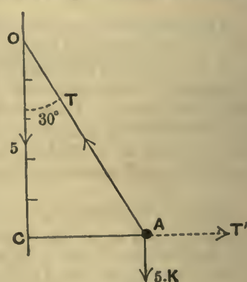


Fig. 16.

Also since the angle at O is 30° we have

$$AC=\tfrac{1}{2}AO;\quad OC=\tfrac{1}{2}\sqrt{3}\,AO,$$

and
$$\frac{5}{OC}=\frac{T'}{CA}=\frac{T}{AO}.$$

$$\therefore\ T=5\,\frac{AO}{OC}=\frac{10}{\sqrt{3}}\text{ kilos weight,}$$

$$T'=5\,\frac{AC}{OC}=\frac{5}{\sqrt{3}}\text{ kilos weight.}$$

(5) *Weights W_1, W_2, W_3 are attached to three points A_1, A_2, A_3 in a string the ends of which are secured to two fixed points A, B. The whole hangs in a vertical plane and the form taken by the string is drawn to scale. W_1 is known. Find the weights of W_2 and W_3 and the tensions of the parts of the string.*

Take a vertical line X_1X_2 (Fig. 17) to represent W_1. From X_1 draw

X_1O parallel to AA_1, and from X_2 draw X_2O parallel to A_1A_2. The three

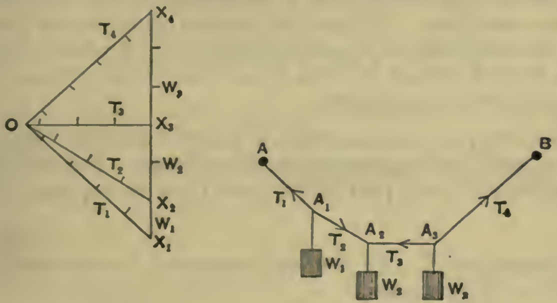


Fig. 17.

forces at A_1 are parallel to the sides of the triangle OX_1X_2, we have therefore

$$\frac{W_1}{X_1X_2}=\frac{T_1}{OX_1}=\frac{T_2}{OX_2}.$$

Thus T_1 and T_2 can be found by measuring OX_1 and OX_2. Draw OX_3 parallel to A_2A_3 meeting X_1X_2 produced in X_3; the three forces T_2, T_3 and W_2 at A_2 are parallel to the sides of the triangle X_2OX_3, they are therefore represented by these sides, thus

$$\frac{T_2}{OX_2}=\frac{T_3}{OX_3}=\frac{W_2}{X_3X_2}.$$

Similarly draw OX_4 parallel to A_3B and produce X_2X_3 to meet OX_4 in X_4. Then T_3, T_4 and W_3 are parallel to X_3O, OX_4 and X_4X_3.

Hence $$\frac{T_3}{X_3O}=\frac{T_4}{OX_4}=\frac{W_3}{X_4X_3}.$$

Thus T_4 and W_3 can be found.

Hence if W_1 be known, the values of W_2 and W_3 together with the tensions in the different parts of the string are given graphically.

In the diagram as drawn it will be found that

$$X_2X_3=2X_1X_2,\quad X_3X_4=3X_1X_2.$$

Hence if $W_1=1$ kilo, then $W_2=2$ kilos, $W_3=3$ kilos.

Also $T_1=4{\cdot}2$ kilos, $T_2=3{\cdot}7$ kilos, $T_3=3{\cdot}3$ kilos,

and $T_4=4{\cdot}6$ kilos.

Hence the tensions and two of the weights are found.

When two or more forces impressed on a particle are given in direction and magnitude it is possible to find expressions for their resultant. We have in the case of two forces to determine the diagonal of a parallelogram two of whose sides are given, while the case of more than two forces involves an extension of the same process.

Thus the propositions given on pp. 33—38 of the Dynamics with regard to displacements apply to forces. For the sake of completeness in this part of the book they are repeated here.

PROPOSITION 6. *To find an expression for the resultant of two forces at right angles.*

Let OA, OB, Fig. 18, represent two forces P, Q respectively at right angles to each other. Complete the rectangle $AOBC$. Let R be the resultant of P and Q, then R is represented by OC.

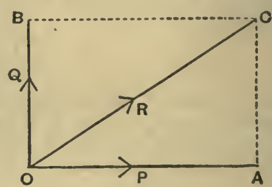


Fig. 18.

Since the angle OAC is a right angle we have

$$OC^2 = OA^2 + AC^2$$
$$= OA^2 + OB^2,$$
$$\therefore\ R^2 = P^2 + Q^2.$$

Hence $$R = \sqrt{\{P^2 + Q^2\}}.$$

PROPOSITION 7. *To find an expression for the resultant of two forces inclined to each other at any angle.*

Let OA, OB represent respectively two forces P, Q inclined to each other at an angle γ.

Complete the parallelogram $AOBC$. OC represents R the resultant of P and Q. Draw CD perpendicular to OA meeting OA produced, Fig. 19 (a), or OA, Fig. 19 (b), in D. Then $AOB = \gamma$; in Fig. 19 (a) the angle γ is less than a right angle; in Fig. 19 (b) it is greater.

Now in Fig. 19 (a),

$$OD = OA + AD = OA + AC \cos DAC$$
$$= OA + OB \cos AOB = P + Q \cos \gamma,$$
$$CD = AC \sin DAC = OB \sin \gamma = Q \sin \gamma.$$

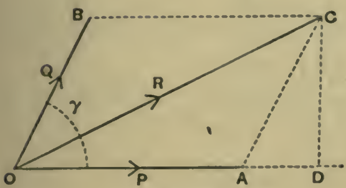


Fig. 19 (a).

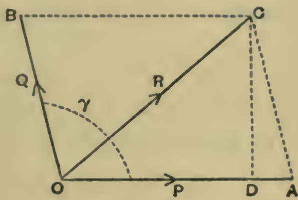


Fig. 19 (b).

In Fig. 19 (b),

$$OD = OA - AD = OA - AC \cos DAC$$
$$= OA - OB \cos (180 - \gamma) = OA + OB \cos \gamma$$
$$= P + Q \cos \gamma,$$
$$CD = AC \sin DAC = OB \sin (180 - \gamma) = Q \sin \gamma.$$

Hence in either case we have

$$R^2 = OC^2 = OD^2 + DC^2$$
$$= (P + Q \cos \gamma)^2 + Q^2 \sin^2 \gamma$$
$$= P^2 + Q^2 + 2PQ \cos \gamma;$$
$$\therefore \ R = \sqrt{\{P^2 + Q^2 + 2PQ \cos \gamma\}}.$$

There are many special cases of this last proposition which can be solved by Geometry without reference to Trigonometry. Thus, suppose the angle between the two forces to be 45°.

Hence, constructing Fig. 20 as above, we have

$$AD^2 + CD^2 = AC^2 = Q^2.$$

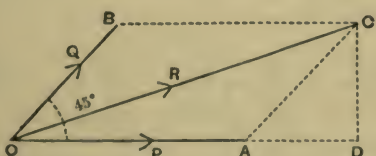


Fig. 20.

Also
$$AD = DC;$$
$$\therefore\ AD = DC = \frac{Q}{\sqrt{2}}.$$

And
$$OD = OA + AD = P + \frac{Q}{\sqrt{2}}.$$

Hence
$$R^2 = OC^2 = OD^2 + DC^2$$
$$= \left(P + \frac{Q}{\sqrt{2}}\right)^2 + \frac{Q^2}{2}$$
$$= P^2 + Q^2 + PQ\sqrt{2}.$$

Or again, if $\gamma = 60°$, we have, fig. 21,
$$AD = \tfrac{1}{2}AC = \tfrac{1}{2}Q,\ \ CD = \frac{Q\sqrt{3}}{2}.$$

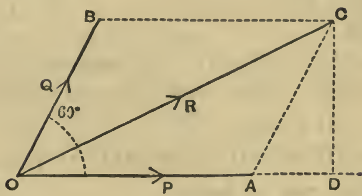


Fig. 21.

$$R^2 = \left(P + \frac{Q}{2}\right)^2 + \frac{3Q^2}{4}$$
$$= P^2 + Q^2 + PQ.$$

These are both given by the general formula by putting $\gamma = 45°$, $\cos\gamma = \sqrt{\tfrac{1}{2}}$ and $\gamma = 60°$, $\cos\gamma = \tfrac{1}{2}$.

If the two forces be equal the resultant bisects the angle between them; for, Fig. 22, if
$$OA = AC,$$
then
$$\angle AOC = \angle ACO$$
$$= \angle BOC.$$

Join AB, cutting OC in D, then AB bisects OC at right angles. And
$$R = OC = 2OD = 2OA\cos AOC$$
$$= 2P\cos\tfrac{1}{2}\gamma.$$

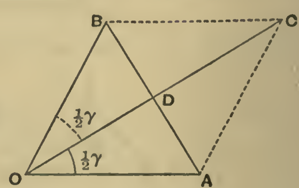


Fig. 22.

9. The Resolution of Forces. Just as we can combine or compound two or more forces and find their resultant, so conversely we can resolve a single force into a number of others, called its components, which are equivalent to it.

PROPOSITION 8. *To find, by a graphical construction, the components of a force in any two directions.*

Let OC, Fig. 23, be the given force, and LM, LN the two given directions. Through O draw OA parallel to LM and through C draw AC parallel to LN. These two forces OA, AC

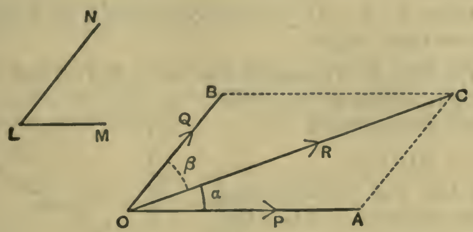


Fig. 23.

acting at O have OC for their resultant, hence OA, AC are components of OC and they are parallel respectively to LM and LN, that is, they are drawn in the given directions.

PROPOSITION 9. *To find an expression for the components of a force in two given directions.*

Let OC, Fig. 23, represent R the given force, and let OA, OB be the components in directions making angles α, β, respectively with OC.

Then $$AOC = \alpha,$$
$$BOC = ACO = \beta.$$

Hence $$OAC = 180 - (\alpha + \beta).$$

Now in the triangle OAC the sides are proportional to the sines of the opposite angles.

Hence $$\frac{OC}{\sin OAC} = \frac{OA}{\sin ACO} = \frac{AC}{\sin AOC},$$

$$\therefore \quad \frac{R}{\sin(\alpha + \beta)} = \frac{P}{\sin \beta} = \frac{Q}{\sin \alpha}.$$

Moreover from the figure $\alpha + \beta = \gamma$.

Hence $$P = R\frac{\sin \beta}{\sin \gamma},$$

$$Q = R\frac{\sin \alpha}{\sin \gamma}.$$

PROPOSITION 10. *To find the components of a force in two directions at right angles.*

Let OC, Fig. 24, represent the force R, OA, OB two directions at right angles in which the components are required.

Let $AOC = \alpha$.

Draw CA, CB perpendicular on the two directions. Then OA, OB represent the components P, Q.

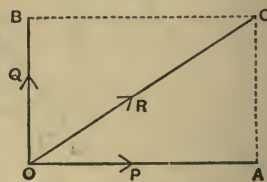


Fig. 24.

Also $\dfrac{OA}{OC} = \cos AOC = \cos \alpha$,

$$\therefore \quad OA = OC \cos \alpha.$$

Hence $$P = R \cos \alpha.$$

Again $\dfrac{OB}{OC} = \cos BOC = \sin AOC = \sin \alpha$,

$\therefore \quad OB = OC \sin \alpha.$

Hence $$Q = R \sin \alpha.$$

If we put $BOC = \beta$ we have clearly

$$OB = OC \cos \beta,$$

$$Q = R \cos \beta.$$

And in this case $\alpha + \beta = 90°$.

Thus, when a force is resolved into two others mutually at right angles, the component in each direction is found by

multiplying the original force by the cosine of the angle between it and the direction of the component.

It must be remembered that this result is only true when the two components are at right angles.

Thus, let OA, OB (Fig. 25) be two components of OC at right angles. Draw OB' making an angle γ with OA' and through C draw CA' parallel to OB'. If now OC be resolved into two forces in directions OA and OB' inclined at an angle γ, the component in the direction OA is no longer OA but OA'.

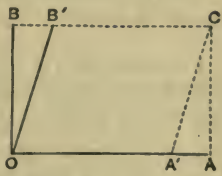


Fig. 25.

A force represented by OA is $R\cos\alpha$, where α is the angle between OA and OC, that represented by OA' has not this value.

Proposition 11. *To shew that if the components of a force be resolved in any direction, then the sum of these resolved parts is equal to the component of the original force resolved in this same direction.*

Take the case of a force R represented by OC, Fig. 26, which is resolved into two forces P, Q represented by OA and OB respectively.

Draw Ox, Oy two lines at right angles through O, and draw AL, BM and CN perpendicular to Ox. Then if we suppose all the forces resolved in the directions Ox and Oy it is clear that OL represents the component of P, OM of Q and ON of R.

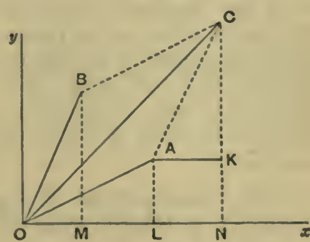


Fig. 26.

Draw AK parallel to Ox to meet CN in K.

Then $LN = AK$. And in the triangles BOM and CAK the sides are respectively parallel and OB is equal to AC.

Hence the triangles are equal. Therefore $OM = AK = LN$. Thus LN represents the component of Q in the direction of Ox.

But from the figure

$$ON = OL + LN = OL + OM.$$

Hence

Component of R in the direction $Ox =$
Component of P + Component of Q.

This proposition can readily be extended to the case of any number of forces.

PROPOSITION 12. *To find an expression for the resultant of a number of forces impressed on a particle in given directions lying in one plane.*

Let P_1, P_2 ... be the forces and let them make angles α_1, α_2 ... with a fixed line Ox, Fig. 27, drawn through the point of action. Let Oy be perpendicular to Ox. Let R be the resultant force and θ the angle its direction makes with Ox.

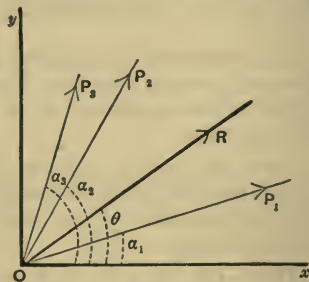


Fig. 27.

Resolve all the forces and the resultant in the two directions Ox and Oy. Then since the resultant is equivalent in its effect to the forces the components of the resultant in each of these two directions are respectively equal to the sum of the components of the forces in these two directions.

The component of the resultant along Ox is $R\cos\theta$, the components of the forces are $P_1\cos\alpha_1\ldots$, $P_2\cos\alpha_2\ldots$ respectively.

Hence

$$R\cos\theta = P_1\cos\alpha_1 + P_2\cos\alpha_2 + \ldots = \Sigma\{P\cos\alpha\} \ldots(1),$$

where $\Sigma\{P\cos\alpha\}$ means the sum of a number of quantities like $P\cos\alpha$.

Again resolving parallel to Oy

$$R\sin\theta = P_1\sin\alpha_1 + P_2\sin\alpha_2 + \ldots = \Sigma\{P\sin\alpha\} \ldots(2).$$

Thus remembering that $\sin^2\theta + \cos^2\theta = 1$, we have by squaring and adding

$$R^2 = [\Sigma\{P\cos\alpha\}]^2 + [\Sigma\{P\sin\alpha\}]^2$$
$$= \Sigma\{P^2\} + 2\Sigma\{P_1P_2\cos(\alpha_2-\alpha_1)\},$$

while by dividing (2) by (1) we find

$$\tan\theta = \frac{\Sigma\{P\sin\alpha\}}{\Sigma\{P\cos\alpha\}}.$$

10. Equilibrium of Forces impressed on a particle. We have already found, Prop. 3, the conditions of equilibrium of a set of forces impressed on a particle. If a polygon be drawn whose sides represent the forces in direction and magnitude it will be closed. The same result can be expressed in symbols by the aid of the last proposition thus.

If the particle be in equilibrium the resultant of the forces is zero. Thus the components of the resultant in any two directions must also be zero. Hence the sum of the components of the forces in any two directions at right angles must be zero.

Hence

$$P_1\cos\alpha_1 + P_2\cos\alpha_2 + \ldots = 0 \text{ or } \Sigma(P\cos\alpha) = 0,$$

and $$P_1\sin\alpha_1 + P_2\sin\alpha_2 + \ldots = 0 \text{ or } \Sigma(P\sin\alpha) = 0.$$

This result is of course applicable to the case of three forces, but in this case we know in addition that the forces are represented by the sides of any triangle drawn parallel to their directions; hence for three forces we have the following theorem known as Lami's Theorem.

PROPOSITION 13. *When three forces impressed on a particle are in equilibrium each is proportional to the sine of the angle between the other two.*

Let the three forces be P, Q, R acting in directions OL, OM, ON respectively, Fig. 28.

Let ABC be a triangle whose sides are parallel to the forces, BC being parallel to OL, CA to OM and AB to ON.

Then from the figure

$$CAB = 180 - MON,$$
$$ABC = 180 - NOL,$$
$$BCA = 180 - LOM.$$

Hence $$\sin CAB = \sin MON,$$
$$\sin ABC = \sin NOL,$$
$$\sin BCA = \sin LOM.$$

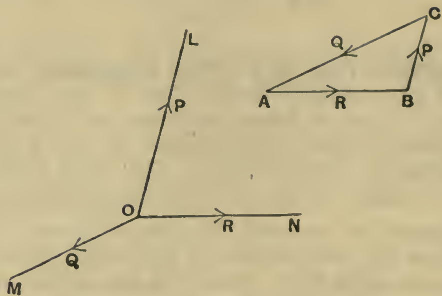


Fig. 28.

But the forces P, Q, R are proportional to the sides BC, CA and AB of the triangle ABC respectively.

Moreover the sides of a triangle are proportional to the sines of the opposite angles. Thus the forces are proportional to the sines of the angles of the triangle which are opposite to them.

Hence $$\frac{P}{\sin A} = \frac{Q}{\sin B} = \frac{R}{\sin C}.$$

Thus $$\frac{P}{\sin MON} = \frac{Q}{\sin NOL} = \frac{R}{\sin LOM}.$$

This theorem has already been verified by experiment (Exp. 1 (*b*).)

We may conclude then that in dealing with questions on the equilibrium of forces impressed on a particle:

(*a*) *The direct method of solution is to resolve the forces in any two convenient directions at right angles and equate to zero each set of components.*

(*b*) *If the forces be only three in number a graphical solution based on the triangle of forces can easily be obtained.*

The following examples will illustrate these various methods.

Examples. (1) *Find the resultant of two forces of* 3 *and* 4 *kilos weight respectively impressed on a particle at right angles.*

Let R be the resultant; then since the forces are at right angles

$$R^2 = 3^2 + 4^2 = 25,$$
$$R = 5 \text{ kilos weight.}$$

This is the result which we verified in Experiment 1.

(2) *Find the resultant of two forces of* 10 *and* 5 *kilos weight acting at an angle of* 60°.

Let R be the resultant.

Then substituting in the formula

$$R^2 = P^2 + Q^2 + 2PQ\cos\gamma,$$

we have

$$R^2 = 5^2\{1 + 2^2 + 2 \times 2 \times \tfrac{1}{2}\}$$
$$= 5^2 \times 7,$$
$$R = 5\sqrt{7} \text{ kilos weight.}$$

(3) *A force of* 10 *kilos weight is resolved into two equal forces mutually at right angles; find these forces.*

Since the components are equal, they are equally inclined to the resultant.

Hence the angle between each of them and the resultant is 45°.

Thus if P and Q be their values

$$P = 10\cos 45° = \frac{10}{\sqrt{2}} \text{ kilos wt.,}$$
$$Q = 10\sin 45° = \frac{10}{\sqrt{2}} \text{ kilos wt.}$$

(4) *A force of* 15 *kilos weight is resolved into two at right angles, the value of one of these is twice that of the other; find the forces.*

Let them be P and Q kilos weight.

Then $$P = 2Q,$$
$$P^2 + Q^2 = 15^2,$$
$$Q^2(1+4) = 15^2,$$
$$Q^2 = 15 \times 3,$$
$$Q = 3\sqrt{5} \text{ kilos weight.}$$

Hence $$P = 6\sqrt{5} \text{ kilos weight.}$$

(5) *The resultant of two forces of* 3 *kilos weight and* 5 *kilos weight is a force of* 7 *kilos weight; find the angle between the two.*

Let γ be the angle, then if R be the resultant of two forces P and Q inclined at an angle γ, we know that

$$R^2 = P^2 + Q^2 + 2PQ\cos\gamma,$$
$$7^2 = 5^2 + 3^2 + 2 \times 5 \times 3 \cos\gamma,$$
$$30\cos\gamma = 49 - 25 - 9 = 15,$$
$$\cos\gamma = \tfrac{1}{2},$$
$$\gamma = 60^\circ.$$

Thus the angle required is 60°.

(6) *Forces equal to the weights of* 6, 7, *and* 8 *lb. are impressed on a particle in directions inclined to each other at* 120°. *Find their resultant.*

Three equal forces at angles of 120° are in equilibrium. The given forces are equivalent to forces respectively of 6 lb., (6+1) lb., and (6+2) lb.

The three forces of 6 lb. are in equilibrium, and may therefore be removed from consideration, and there are left forces of 1 and 2 lb. weight at an angle of 120°. Their resultant may be found graphically, or thus

$$R^2 = 1^2 + 2^2 + 2 \times 1 \times 2 \times \cos 120^\circ$$
$$= 5 - 2 = 3.$$

Thus $$R = \sqrt{3} \text{ lb. wt.}$$

Or again.

Let R be the resultant force and let it make an angle θ with the direction of the force of 8 lb. weight.

Resolve all the forces parallel and perpendicular to the direction of the 8 lb. force.

$$R\cos\theta = 8\cos 0 + 7\cos 120 + 6\cos 120$$
$$= 8 \,.\, 1 + 7\left(-\tfrac{1}{2}\right) + 6\left(-\tfrac{1}{2}\right)$$
$$= 8 - 3\tfrac{1}{2} - 3 = 1\tfrac{1}{2}.$$
$$R\sin\theta = 8\sin 0 + 7\sin 120 - 6\sin 120$$
$$= 8.0 + 7\frac{\sqrt{3}}{2} - 6\frac{\sqrt{3}}{2}$$
$$= \frac{\sqrt{3}}{2}.$$
$$\therefore\ R^2 = \frac{12}{4} = 3.$$

Hence $$R = \sqrt{3} \text{ lb. weight.}$$

$$\tan\theta = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}.$$
$$\therefore\ \theta = 30^\circ.$$

Thus the resultant is a force of $\sqrt{3}$ lb. weight inclined at 30° to the force of 8 lb. weight.

(7) *A body weighing 5 lb. is supported by two strings, the tension in one string is 8 lb. weight and its direction is inclined at 30° to the horizon. Find the direction of the other string and the tension in it.*

(i) *Graphically.*

Draw AB (Fig. 29) vertically down, 5 cm. in length, to represent the weight, draw BC 8 cm. in length and at 30° to the horizon to represent the tension in the first string, and join CA. Then CA represents the direction of the second string, and the number of cm. in CA measures in lb. weight the tension in that string.

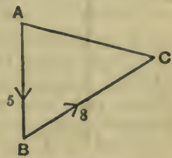


Fig. 29.

(ii) *By resolution of forces.*

Let T be the tension of the string OC, θ the angle it makes with the vertical.

Let OD be the first string and OA the direction of the 5 lb. weight.

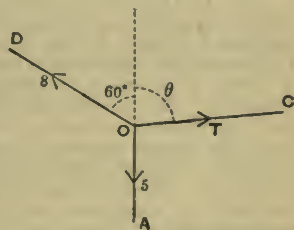


Fig. 30.

Resolving vertically

$$5 = T\cos\theta + 8\cos 60°.$$

Resolving horizontally

$$T\sin\theta = 8\sin 60°.$$

From the first equation

$$T\cos\theta = 1.$$

From the second

$$T\sin\theta = 4\sqrt{3}.$$

Hence

$$T^2 = 49,$$

$$T = 7 \text{ lb. weight},$$

and $$\tan\theta = 4\sqrt{3}.$$

(8) *The resultant R of two forces P, Q impressed on a particle is equal to P and at right angles to it. Find the force Q.*

Let AB (Fig. 31) represent P, BC equal to AB and at right angles to it will represent R. Join CA. Then BC is the resultant of forces represented in magnitude and direction by AB and AC acting at B.

Thus a line through B parallel and equal to AC will represent Q.

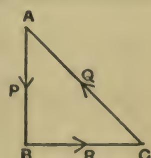


Fig. 31.

But $$AC^2 = AB^2 + BC^2$$

$$= 2AB^2.$$

Therefore $$Q^2 = 2P^2.$$

Hence $$Q = P\sqrt{2}.$$

(9) *Two forces impressed on a particle are represented respectively by* λOA *and* μOB, *A and B being fixed points and* λ *and* μ *constants.*

C is a point in AB such that λAC *is equal to* μBC. *Shew that the resultant of the forces is* $(\lambda+\mu)\,OC$.

By the triangle of forces a force λOA along OA (Fig. 32) is equivalent to λOC along OC and λCA acting at O parallel to CA. Again μOB is equivalent to μOC along OC and μCB at O parallel to CB.

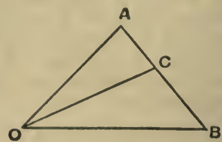


Fig. 32.

Thus the two given forces are equivalent to $(\lambda+\mu)\,OC$ along OC together with λCA and μBC in opposite directions parallel to AB. These last two are equal, they therefore balance and may be removed. Hence the resultant is $(\lambda+\mu)\,OC$.

(10) *Explain the action of the wind in propelling a ship.*

Let AB (Fig. 33) represent the direction of the ship's keel; CD the direction of the sail which we suppose to be flat. Let the pressure of the wind be equivalent to a force P acting on the sail in the direction indicated. Resolve this force into two components, one R at right angles to the sail, the other T along the sail. This last component produces little or no effect, and we may neglect it; it is only the component perpendicular to the sail which we need to consider. This force R acts on the ship through the mast. We may resolve R into two components, the one X parallel to the keel, the other Y at right angles to the keel.

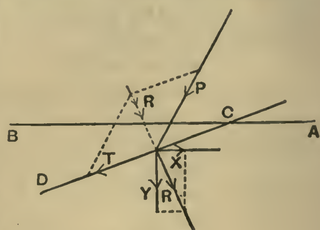


Fig. 33.

The resistance offered by the water to motion in a direction at right angles to the keel is so great that the component Y is almost balanced by it; the ship is built so that the water may offer a small resistance to motion parallel to the keel, and the ship moves in this direction under the impressed force X. The effect of the force in tending to turn the ship round can only be considered later.

EXAMPLES.

TRIANGLE AND PARALLELOGRAM OF FORCES.

1. Forces represented by the weights of 10 lb. and 15 lb. respectively act at a point in northerly and easterly directions. Find the magnitude and direction of their resultant.

2. Find the magnitude of the resultant of two forces $12P$ and $5P$ when they act at a point and in directions at right angles to one another.

3. ABC is a triangle, D, E, and F the middle points of BC, CA, AB respectively. Forces acting at a point are represented in direction and magnitude by the lines AB, AC, BE; shew that their resultant will be similarly represented by $3FD$.

4. The sum of two forces is 36 lb. wt. and the resultant which is at right angles to the smaller of the two is 24 lb. wt. Find the magnitude of the forces.

5. The difference of two forces is 8 lb. wt. and the resultant, which is at right angles to the smaller of the two, is 12 lb. wt. Find the magnitude of the forces.

6. Can three forces which are in the proportion of 7, 10 and 17 keep a point at rest?

7. Forces represented in magnitude and direction by the diagonals of a parallelogram act at one of the corners, what single force will counteract them?

8. Shew by means of a diagram how to find the part of the pressure of the wind which is available for urging on a ship when the wind blows very nearly from the direction in which the ship is going.

9. Explain how the law of composition of forces may be deduced from the second law of motion.

10. Two forces P and Q have a resultant R equal to P. Draw a diagram representing such a system and shew by means of it that the resultant of two forces equal and parallel to P and R respectively would act at right angles to Q.

11. Shew how to find the resultant of two forces acting at the same point and explain how to verify the result by experiment.

12. The wind is blowing from the north-east. Explain with a diagram its action in propelling a ship towards the north.

13. Resolve a force of 15 lb. wt. into two forces each making with it an angle of 30°.

14. Shew how to place three forces which are in the ratio of 3, 4 and 5, so that they may keep a particle at rest.

15. Describe an experiment to prove the parallelogram of forces

16. A pendulum consisting of a bob weighing 1 kilogramme at the end of a string 1 mètre long is drawn aside until the bob is 25 cm. from the vertical through the point of support, and is held in this position by a horizontal string.

Find the forces on the bob (1) when in this position, (2) just after the horizontal string is cut, (3) as the bob swings through its lowest position.

17. Five forces each equal to P act along radii of a circle which are at angular distances 30°, 60°, 90°, 120° and 150° from a fixed radius; determine the resultant.

18. If a heavy body is supported by two strings one of which is vertical, prove that the other must be vertical also.

19. Shew that a vessel may sail due east against a south-east wind.

20. A straight line is drawn parallel to the base BC of a triangle ABC, cutting AB in a point D such that AD is twice BD. If P be any point on this line, prove that the resultant of forces completely represented by AP, BP, CP is parallel to BC.

21. Find the resultant of two forces represented by the side of an equilateral triangle and the perpendicular on this side from the opposite angle.

22. If the magnitude of one of two forces acting at a point be double that of the other, shew that the angle between its direction and that of their resultant is not greater than thirty degrees.

23. If a uniform heavy bar is supported in a horizontal position by a string slung over a peg and attached to both ends of the bar, prove that the tension of the string will be diminished if its length be increased.

24. A weight hangs at the end of a string attached to a peg. If the weight is held aside by a horizontal force so that the string makes an angle of 30° with the vertical, shew that the tension of the string is double the horizontal force.

25. $ABCD$ is a rhombus. Shew, without assuming the Parallelogram of Forces that forces represented by AB, CB, CD and AD are in equilibrium.

26. State how three equal forces must act so as to produce equilibrium and hence find the resultant of two equal forces inclined at 120° to each other.

27. A weight is suspended by means of two equal strings attached to points in the same horizontal line. Shew that if the lengths of the strings are increased, their tension is diminished.

28. A weight hangs at the end of a string attached to a peg. If the weight is held aside by a horizontal force, so that the string makes an angle 60° with the vertical, shew that the tension of the string is double the weight.

29. Two forces P and Q act at the same point and their directions are inclined to each other at an angle of 45°. Find an expression for the magnitude of their resultant.

Find approximately the magnitude of the resultant if the component forces be respectively equal to weights of 3 lb. wt. and 4 lb. wt.

30. If a heavy uniform bar is supported in a horizontal position by a string slung over a peg and attached to both ends of the bar, prove that the tension of the string will be diminished if its length be increased.

31. A weight is suspended by means of two equal strings attached to points in the same horizontal line. Shew that if the distance between the points is increased, the tension of the strings is increased.

32. Find the resultant of two forces of 5 and 10 lb. weight respectively acting at an angle of 60°.

33. A body is acted upon by two forces, one of 500 dynes due north, and one of 250 dynes north-east; find the resultant force.

34. A mass of 1 lb. is supported by strings of lengths 3 and 4 feet respectively attached to two points in the same horizontal plane 5 feet apart. What is the tension of each string?

35. Prove that when a kite is being flown, the position which the string will take up cannot be at right angles to the body of the kite, but will be less steep.

36. Shew that three forces of 5, 6, and 12 lb. wt. can never be in equilibrium.

37. A body weighing 4 lb. at rest on a smooth table is acted upon by forces of 3 and 4 lb. weight in directions oblique to the table and at right angles to each other. Shew by a diagram how to find the directions of the forces.

38. Three forces keep a particle in equilibrium, one acts towards the east, another towards the north-west, and the third towards the south; if the first be 5, find the other two.

39. $ABCD$ is a parallelogram, and three forces acting at a point are represented in magnitude and direction by AC, BD and $2DA$. Shew that the three forces are in equilibrium.

40. DC and AB are diameters of a circle. Three forces acting at a point are represented in magnitude and direction by AB, DC and $2BD$; shew that they are in equilibrium.

41. Three forces of 5, 12 and 13 lb. wt. are in equilibrium. Shew that two of them are at right angles and find the sines of the angles which the remaining force makes with these two.

42. A weight of 10 lb. is suspended from a fixed point by a string 25 inches in length. The weight is drawn aside until its vertical distance below a horizontal line drawn through the fixed point is 20 inches. Shew that the smallest force which will keep the weight in this position will just support a weight of 6 lb. hanging freely.

43. A force of 7 lb. wt. acts on a particle due north, one of 8 lb. wt. due east, one of 6 lb. wt. N.N.W. Find by a careful drawing the direction and the magnitude of the force which will balance these.

44. If two forces each equal to 1 lb. wt. act at a point and their directions make with each other an angle of 60°, find to the nearest oz. the magnitude of their resultant.

45. Shew that forces of 99 lb. wt. and 5 lb. wt. acting at right angles to each other have a resultant which is 17 times as great as the resultant of 5 lb. wt. and 3 lb. wt. acting at right angles to each other.

46. ABC is an equilateral triangle, AB the perpendicular on BC. Forces each equal to P act along BA, AD and AC respectively, in the directions indicated by the letters. Find the magnitude of their resultant, and shew that it is inclined at an angle 75° to the line AB.

47. Three lines AB, AC and AD in the same plane make the angles BAC, CAD each equal to 30°. Forces, equal to P, act along BA, AC and AD in the directions indicated by the letters. Find the magnitude of their resultant, and shew that it is inclined at an angle 75° to the line AB.

48. $ABCD$ is a rectangle, $AD=15$ inches and $AB=20$ inches; find the magnitude of the resultant of two forces of 16 lb. wt. and 25 lb. wt. acting along AB and AC respectively.

49. To two points A, B, 5 feet apart on a horizontal beam, the ends of a string ACB are attached, AC being 4 feet and BC 3 feet long. From C a weight of 10 lb. is hung. Find the tensions in the strings AC and BC.

50. Four weights of 2, 3, 4 and 5 lb. are hung on a string 5 feet long at points 1 foot apart. The ends of the string are attached to two points 3 feet apart in the same horizontal line, and the form assumed by the string is drawn to scale on a sheet of paper. Shew how to find from the figure the tensions in each part of the string.

CHAPTER II.

PARALLEL FORCES.

11. Rigid Bodies. So far we have dealt only with forces impressed on a particle, or on some body which for our purpose could be treated as a particle. We are to consider now some of the effects of forces impressed on a body, the volume of which cannot be treated as very small. The bodies with which we shall deal are called **Rigid Bodies.** By this it is meant that they do not alter in shape when force is impressed. No body is perfectly rigid, but many substances will resist the application of force and will change in shape by an amount which is practically infinitesimal, when force is applied. Iron, glass and wood have all rigidity, and, though the shape of a body of any of these and other similar materials may vary slightly under the application of a force, the variation will not concern us. Bodies which have rigidity are called **Solids.** Other bodies which we consider in Hydrostatics are **Fluids.** The distinction between Solids and Fluids will best be considered later[1].

12. Superposition of Forces. Now, when a force is impressed on a rigid body, it is found by experiment that any point of the body, which lies in the line of action of the force, may be considered as the point of application of the force.

Thus if a body be in equilibrium under two forces P and Q applied at A and B, Fig. 34, the two forces P and Q must be equal and act in opposite directions along the line AB.

Fig. 34.

Now let a force P act on the body at

[1] See Hydrostatics.

A in direction AB, Fig. 35. At B introduce two equal and opposite forces each equal to P. These forces are in equilibrium and will therefore not disturb the effect of P. Then P at A and P at B directed along BA balance; they therefore produce no effect and may be removed from consideration, and we are then left with P at B in the direction AB producing the same effect as P at A impressed in that same direction.

Fig. 35.

Thus P may be impressed at any point of the body in its line of action without altering its effect.

This is called the principle of the transmissibility of force.

In proving this principle, as well as in some of the Examples solved in Chapter I., we have made use of another principle which is of general application. We have introduced two equal forces in opposite directions and have assumed that this does not affect the equilibrium. The truth of the assumption is obvious.

Thus we may, without altering the conditions of any problem, superpose upon, or remove from, any system of forces any second system which is itself in equilibrium.

13. Parallel Forces. When the lines of action of two or more forces impressed at two or more points in a body meet, we may suppose the forces to be applied at the point of intersection of their lines of action, and find their resultant by the rules established in the last chapter.

In general, however, the lines of action of forces impressed on a body do not all meet in a point; the problem is more complicated. The simplest case of this occurs when the forces are parallel.

DEFINITION. *Two parallel forces are said to be* Like *when they act in the same direction, they are* Unlike *when they act in opposite directions.*

PROPOSITION 13 A. *To find the resultant of two parallel forces impressed at two points of a rigid body.*

(i) *When the forces are like.*

Let the two forces be P, Q acting at the points A, B, Fig. 36,

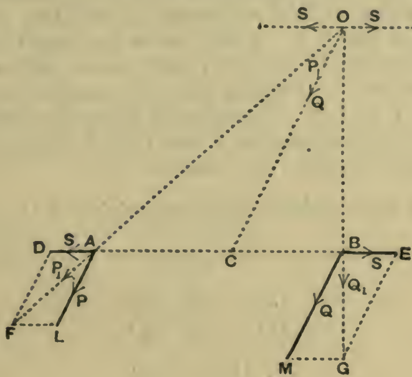


Fig. 36.

in directions AL and BM. Take AL and BM to represent the forces. Join AB, and at A and B introduce two equal and opposite forces S, represented respectively by AD and BE acting in the directions AD and BE. Complete the parallelograms $ADFL$ and $BEGM$.

The resultant of P and S at A is represented by AF and acts along AF, let it be P_1 and replace P and S by their resultant.

The resultant of Q and S at B is expressed by BG, let it be Q_1 acting along BG.

Then the two forces P and Q are equivalent to P_1 and Q_1 impressed at A and B and represented by AF and BG respectively.

The lines of action P_1 and Q_1 when produced will meet. Let them be produced and meet at O. Draw OC parallel to the original direction of the forces P and Q to meet AB in C. Transfer the points of application of P_1 and Q_1 from A and B respectively to O.

At O resolve P_1 into two components parallel to OC and CA respectively, these components must be equal to P and S.

Again resolve Q_1 at O into two components parallel to OC and CB, these two components will be Q and S. Thus we now have, acting at O, $P+Q$ along OC and two equal forces S parallel to CA and CB. These forces are equal and opposite, they may therefore be removed and we are left with a single force R equal to $P+Q$ acting along OC parallel to the original direction of P and Q. Transfer the point of application of R to C. Then the resultant of P and Q is R acting at C parallel to the original direction of P and Q.

We have now to determine the position of the point C.

By the construction, the triangles ADF and ACO are similar.

Hence $$\frac{OC}{CA} = \frac{FD}{DA} = \frac{P}{S},$$

or $$P \,.\, AC = S \,.\, OC.$$

The triangles BEG and BCO are similar.

Thus $$\frac{OC}{CB} = \frac{GE}{EB} = \frac{Q}{S},$$

or $$Q \,.\, BC = S \,.\, OC.$$

Thus $$Q \,.\, BC = P \,.\, AC,$$

or $$\frac{AC}{BC} = \frac{Q}{P}.$$

Again add unity to each side

$$\frac{AC}{BC} + 1 = \frac{Q}{P} + 1,$$

$$\frac{AC+BC}{BC} = \frac{P+Q}{P} = \frac{R}{P},$$

or $$P \,.\, AB = R \,.\, BC.$$

Similarly $$Q \,.\, AB = R \,.\, AC.$$

Thus $$\frac{P}{BC} = \frac{Q}{AC} = \frac{R}{AB}.$$

(ii) *When the forces are unlike.*

Let the forces be P and Q acting in the opposite directions AL, BM, Fig. 37, at the points A and B; let P be the greater force.

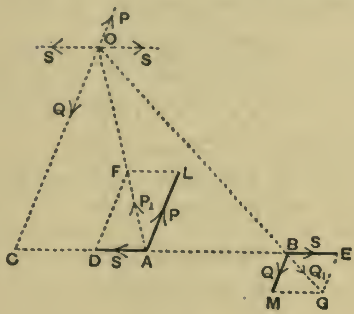


Fig. 37.

Take AL and BM to represent the forces. Join AB, and at A and B introduce two equal and opposite forces S represented respectively by AD and BE impressed in the directions AD and BE.

Complete the parallelograms $ADFL$ and $BMGE$ as before.

Replace P and S, acting at A, by their resultant P_1 along the diagonal AF, replace Q and S at B by their resultant Q_1 along BG.

Since Q is less than P the two diagonals AF and GB must meet. Let them meet at O; draw OC parallel to the original directions of the forces to meet BA produced in C. Remove the points of application of P_1 and Q_1 from A and B respectively to O.

At O resolve P_1 into its two components P and S parallel to AL and AD respectively, and resolve Q_1 into its two components Q and S parallel to BM and BE respectively.

The two components S balance and may be removed, and we have left a force R equal to $P - Q$, in the direction CO pro-

duced, which is equivalent to the two forces P and Q at A and B. Transfer the point of application of R to C. Then the resultant required is R equal to $P - Q$ acting at C.

To determine the position of C we have, as before, since the triangles ADF and ACO are similar,

$$\frac{OC}{CA} = \frac{FD}{DA} = \frac{P}{S}.$$

Thus $$P.AC = S.OC.$$

Also, since BEG and BCO are similar,

$$\frac{OC}{CB} = \frac{GE}{EB} = \frac{Q}{S}.$$

Thus $$Q.BC = S.OC.$$

Hence $$P.AC = Q.BC,$$

or $$\frac{P}{BC} = \frac{Q}{CA} = \frac{P-Q}{BC-CA} = \frac{R}{AB}.$$

Thus for like forces we have

$$R = P + Q,$$

for unlike forces

$$R = P - Q,$$

while in either case the position of the point C is given by

$$\frac{P}{BC} = \frac{Q}{CA} = \frac{R}{AB}.$$

14. Couples. We should notice that there is one case of the last proposition (Proposition 13 A ii.) in which the construction fails; if P is equal to Q the lines AF and GB of Fig. 37 will be parallel; we cannot find their point of intersection, it is at an infinite distance away and there is no single force which will replace the two; such a system of two equal unlike parallel forces is called a couple.

We shall consider such a system later.

15. Resultant of Paıallel Forces. When a number of parallel forces are impressed on a body we can find their

resultant by taking two and finding the magnitude and line of action of their resultant; then combine with this resultant a third force and so on, in this way the resultant of all the forces can be obtained. It is clearly a force R given by

$$R = P_1 + P_2 + P_3 + \ldots = \Sigma\{P\},$$

where $P_1, P_2 \ldots$ are the individual forces all supposed to act in the same direction; if one or more of the forces acts in the opposite direction its sign, in the expression for R, must be changed.

16. Experiments on Parallel Forces. For experiments on parallel forces a rectangular bar of wood about a metre long with square section, each side of the square being some 2 to 3 cm. in length, is convenient.

One face of the bar should be graduated in centimetres or inches as in Fig. 38.

Three rings of brass wire A, B, C are made to fit the bar

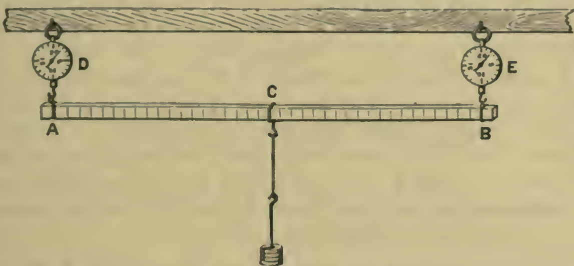


Fig. 38.

and can slide along it. These rings have small hooks attached as shewn in the figure. The bar is suspended in a horizontal position by vertical strings attached to A and B while various weights can be supported from C.

The ends of the strings attached to A and B are secured to two spring balances D and E.

For this experiment Salter's circular balances are the most convenient. In a spiral balance the scale-pan drops consider-

ably when loaded, owing to the stretching of the spring; the drop is much less in the circular form shewn in the figure.

The small motion of the end of the spring is magnified by the pointer.

The spring balances are supported in some convenient way and the bar suspended in a horizontal position.

If the bar be uniform and the rings A, B be equidistant from either end respectively, it will be found that with the central ring C unloaded each balance is equally stretched. This extension is due to the weight of the bar; in making experiments the readings of the balances thus obtained should be subtracted from those observed when the bar is loaded.

EXPERIMENT 2. *To determine by experiment the position and magnitude of the resultant of two parallel forces.*

Suspend the bar as just described from the two balances D and E by the rings A, B. Adjust the length of the strings so that the bar may be horizontal and the rings A, B equidistant from its centre. Take the readings of the spring balances, these should be the same. Suppose them to be ·25 kilo. Suspend a carrier or scale-pan of known weight from C, and place known weights on it. Let the total weight suspended from C be R kilos weight, this is supported by the tensions in the two strings at A and B. Read the spring balances and subtract from each the reading observed before R was suspended. The differences give the values of the forces P, Q which acting at A and B respectively support R.

Note the positions on the scale of the points A, B and C and thus measure the distances AC and BC.

Then it will be found that

$$R = P + Q,$$

and also that

$$P \,.\, AC = Q \times BC.$$

The resultant of P and Q is clearly a force through C equal and opposite to R, thus the formula which gives the magnitude and line of action of the resultant of two parallel forces is verified.

Repeat the experiment; shifting the position of C it will be found that as C is moved P and Q both change, but their sum remains constant while they always satisfy the relation

$$P \,.\, AC = Q \,.\, BC.$$

Moreover if the value of R be altered by changing the suspended weights it will be found that the values of P and Q are also changed but in such a way that the equation $P + Q = R$ is always true.

Again if O be a point on the bar on the side of A removed from B, and the distances OA, OB and OC be measured, it will be found that

$$R \,.\, OC = P \,.\, OA + Q \,.\, OB.$$

See Section 21.

17. Motion about an axis. We will now consider the equilibrium of a body which can turn round a fixed axis, and on which forces are impressed in a plane perpendicular to the axis. Unless some relation exists between these forces rotation will take place, we wish to determine the relation which must exist if there is to be equilibrium. This condition is easily found; the forces, which we have to consider, are the impressed forces and the reaction at the axis; this reaction is a force which necessarily passes through the axis, and it must balance the resultant of the impressed forces. Hence for equilibrium the resultant of the impressed forces must pass through the axis.

Suppose now that there are only two impressed forces. We can put the conditions into symbols thus.

PROPOSITION 14. *To find the condition of equilibrium of a body which can turn about a fixed axis when acted on by two forces in a plane perpendicular to the axis.*

It is clear from what has been said that the resultant of the forces must pass through the axis.

Let the forces P, Q act in the plane of the paper and let the axis, at right angles to that plane, cut it in C. Let a line ABC through the axis cut the lines of action of the forces in A and B and transfer the points of application of the forces to A and B; then the resultant of two forces P and Q acting at A and B respectively passes through C.

(i) Let the forces be parallel, Fig. 39.

Then their resultant R, which is equal to $P+Q$, passes through C and we have

$$P \,.\, AC = Q \,.\, BC.$$

Draw DCE perpendicular to the forces to meet their lines of action in D and E and let $CD = p$, $CE = q$.

The triangles ACD, BCE are similar.

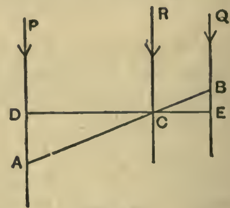


Fig. 39.

Hence $\dfrac{p}{q} = \dfrac{CD}{CE} = \dfrac{CA}{CB} = \dfrac{Q}{P}$.

Therefore $P \,.\, p = Q \,.\, q$.

Now p and q are the perpendiculars from the axis on the lines of action of P and Q. Thus in this case the condition of equilibrium is that

$P \times$ perpendicular from axis $= Q \times$ perpendicular from axis.

(ii) Let the forces P, Q, Fig. 40, be not parallel but let their lines of action meet at O. Their resultant R acts through O hence R acts along OC. Take OC to represent the resultant and from C draw CF, CG parallel to BO and AO respectively to meet OA and OB in F and G. Draw CD and CE perpendicular to the lines of action of P and Q, and let their lengths be p and q. Then $OFCG$ is a parallelogram and OC represents the resultant of the forces P, Q along OF and OG respectively.

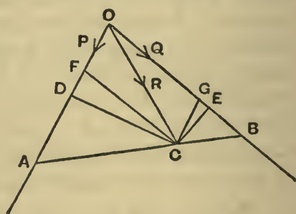


Fig. 40.

Thus OF represents P and OG represents Q.

Now the diagonal of a parallelogram bisects it.

Hence area of triangle OFC = area of triangle OGC,

and area $OFC = \frac{1}{2} OF \,.\, CD$,

also area $OGC = \frac{1}{2} OG \,.\, CE$.

Thus $$OF\,.\,CD = OG\,.\,CE.$$

Hence $$\frac{p}{q} = \frac{CD}{CE} = \frac{OG}{OF} = \frac{Q}{P}.$$

Therefore $$P\,.\,p = Q\,.\,q.$$

Thus the condition of equilibrium is the same as before,

$P \times$ perpendicular from axis $= Q \times$ perpendicular from axis.

It may be shewn that, when three or more forces act, the condition of equilibrium is the same in form; the quantities involved are the strength of each force multiplied by the length of the perpendicular from the point C on the line of action of the force.

18. Moment of a Force. The quantity which we have thus been led to consider has been given a name; it is called the moment of the force about the point.

DEFINITION. *The* Moment *of a force about a given point is the product of the force and the perpendicular drawn from the point on to the line of action of the force.*

Thus the condition of equilibrium just found may be expressed by the statement that *The moment of P round C is equal to the moment of Q round C.*

Again if we consider the force P only it will turn the body in one direction round the axis, while Q, alone, would turn it in the opposite direction. Under the action of P the body would rotate in a direction opposite to that of the hands of a watch, placed face uppermost on the page, under the action of Q it would rotate in the same direction as the hands of the watch. This fact is generally expressed by the statement that the moments about C of P and Q are opposite in sign.

The moment of P is said to be positive, that of Q is negative.

Thus when the bar is in equilibrium the moments of the forces round C are equal in magnitude and opposite in sign, or in other words, having regard to the difference in sign, we may state that the sum of the moments about C of all the forces is zero.

19. Experiments on Moments. We shall now describe some experiments to verify this result.

EXPERIMENT 3. *To prove that, when a body which can turn about a fixed point is in equilibrium under two forces, the moments of these forces about the point are equal and opposite.*

For this experiment the bar employed in Experiment 2 is again used. It is however suspended by an axis through C, Fig. 41, so that it can rotate in a vertical plane.

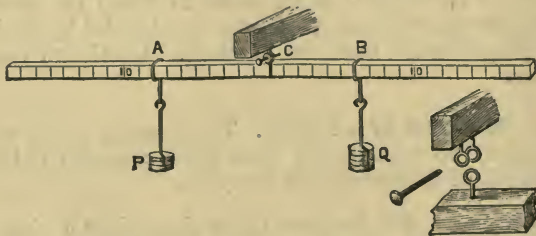


Fig. 41.

This is attained either by boring a hole through the centre of the bar and passing a piece of steel wire through the hole, the ends of the wire pass through two screw-eyes fixed into some convenient support, the wire thus forms the axis; or if more convenient, a screw-eye may be fixed into the bar instead of boring a hole through it. A pin run through these eyes forms the axis. By suspending the bar very close to its centre, the effect of its own weight in tending to turn it will be very small, and may be omitted from consideration. [See §§ 35, 37 *Centre of Gravity.*]

(i) *When the forces are parallel.*

Suspend by means of the wire rings carriers from two points A, B of the bar. Let C be the fixed point or fulcrum. Place weights on the carriers until the bar balances in a horizontal position. Let P be the total weight, including that of the carrier, suspended from A, Q that suspended from B.

Determine the weights P, Q for various positions of the carriers A, B and measure in each case the distances AC and BC. Form a table giving in four columns corresponding values of P, Q, AC and BC, then calculate the products $P \,.\, AC$ and $Q \,.\, BC$.

It will be found in all cases that they are equal, moreover A and B are on opposite sides of C, hence the moments are opposite in sign. Thus the proposition is verified for two parallel forces.

It can be verified for more than two such forces by placing several carriers on the bar and loading until equilibrium is secured.

In all cases it will be found that the sum of the moments about C, each taken with its proper sign, is zero.

(ii) *When the forces P, Q are not parallel.*
The bar is supported as before and two spring balances are suspended from a point O as shewn in Fig. 42. The hooks of

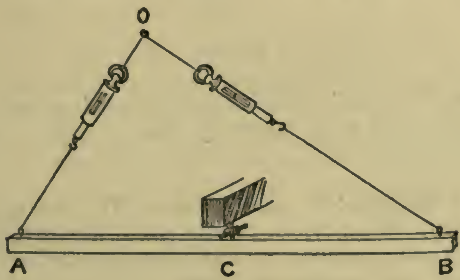


Fig. 42.

the balances are connected by strings to the points A and B and the lengths of the strings are adjusted until the bar is horizontal; the readings of the balances give the tensions P and Q; the perpendicular distances p and q of C from the lines AO and BO are measured, and it will be found as before that

$$P \cdot p = Q \cdot q.$$

In this experiment some small error may be produced by the weights of the balances themselves, but it will not be large if the balances are stretched so that the tension in the strings may be considerable.

Another arrangement of this apparatus is shewn in Fig. 43. The bar is used as before, the strings from A and B pass over pulleys and carry weights. Thus the forces P, Q impressed at

A and B are measured by the weights suspended at the ends of the string, the pulleys which should move easily merely serve

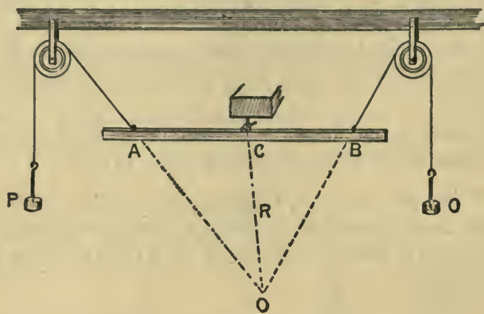


Fig. 43.

to alter the direction in which the forces are impressed. In all these experiments it is necessary that the bar should turn easily on its axis.

20. Principle of the Lever. We have thus verified the law that: When forces in one plane are impressed on a body, which can turn about an axis perpendicular to that plane, and maintain it in equilibrium, the resultant passes through the axis and the sum of the moments of the forces, each taken with its proper sign, about the axis is zero. This law when applied to the case of two forces is often spoken of as the Principle of the Lever.

21. Moments. The moment of a force can be represented geometrically and the representation will be found useful.

Thus let AB, Fig. 44, represent a force P, and let C be a point, about which the moment of P is required. Draw CD perpendicular to AB, Fig. 44 (a), or to AB produced, Fig. 44 (b), and join AC and BC. Then the moment of P about C is equal to $P . CD$.

But P is represented by AB.

Hence $P \, . \, CD$ is represented by $AB \, . \, CD$.

And $AB \, . \, CD = 2$ area triangle ABC.

Thus the moment of a force is twice the area of a

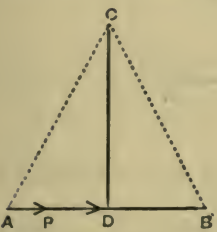


Fig. 44 (*a*).

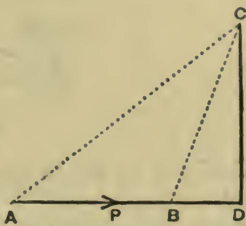


Fig. 44 (*b*).

triangle whose base is the line representing the force completely, and vertex the point round which the moment is being taken.

It should be noticed, that in making use of this proposition, the line AB must represent the force *completely*. It must therefore pass through the point of application of the force. Suppose OA, OB be two lines representing forces P, Q impressed on a particle at O; complete the parallelogram $AOBC$, then for some purposes, e.g. in order to find the resultant, we may treat AC, which is equal and parallel to OB as representing Q, when so doing we bear in mind all the time that it represents it only in magnitude and direction, NOT in point of application, we cannot calculate the moment of Q about some point such as D, by finding the area of the triangle DCA we must find the area of DOB.

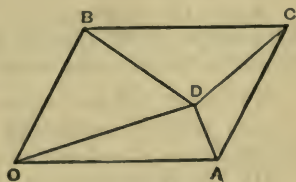


Fig. 45.

We proceed now to some theorems about moments which are of great importance.

PROPOSITION 15. *To prove that the algebraic sum of the moments of two forces about a point in their plane is equal to the moment of their resultant about that point.*

(i) *When the lines of action of the forces meet.*

Let the forces be P, Q impressed on a particle at O in directions OA, OB respectively.

Let H, Fig. 46, be the point about which the moments are

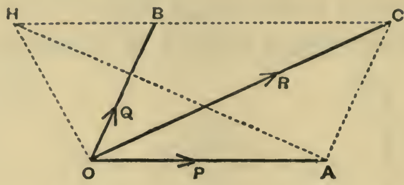


Fig. 46.

required and let OC be the direction of R, the resultant of P and Q.

Draw HC, parallel to OA, to meet OC in C, and OB in B, and from C draw CA parallel to BO to meet OA in A.

Then $AOBC$ is a parallelogram having OC for its diagonal. Take OC to represent the resultant R, then OA and OB must represent P and Q the components of R in the directions OA and OB respectively.

Thus the forces P, Q and R are represented respectively by OA, OB and OC.

Again the moment of P about H is represented by twice the triangle HOA, that of Q by twice the triangle HOB, and that of R by twice the triangle HOC.

Two cases will now arise.

(*a*) If H is outside the angle AOB, Fig. 46, the moments of both forces P and Q about H are the same in sign.

(*b*) If H is within the angle AOB, Fig. 47, the moments of the two forces have opposite signs.

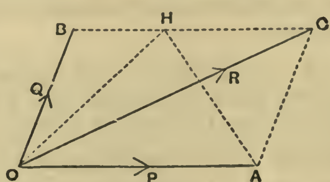


Fig. 47.

In (a)

$$\text{Moment of } R = 2\triangle HOC.$$

Now $$\triangle HOC = \triangle HOB + \triangle COB$$
$$= \triangle HOB + \triangle COA$$
$$= \triangle HOB + \triangle HOA,$$

for HC is parallel to OA.

Thus the moment of R = moment of P + moment of Q.

Again (b)

$$\text{Moment of } R = 2\triangle HOC.$$

Now $$\triangle HOC = \triangle COB - \triangle HOB$$
$$= \triangle COA - \triangle HOB$$
$$= \triangle HOA - \triangle HOB,$$

for BC is parallel to OA.

Moment of R = moment of P − moment of Q.

In either case *The moment of the resultant of two forces is the algebraic sum of the moments of the forces.*

(ii) *When the lines of action of the forces are parallel.*

Let P, Q be the forces and R their resultant. From H, Fig. 48, the point about which the moments are taken, draw $HACB$ perpendicular to the lines of action of the forces meeting them in A, B and C respectively.

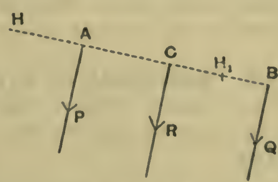


Fig. 48.

In the figure the moments of P and Q about H have the same sign.

Moreover we know that since R is the resultant of P and Q,

$$R = P + Q,$$

and $$P \, . \, AC = Q \, . \, BC.$$

Now the moment of R about H

$$\begin{aligned} &= R\,.\,HC = (P+Q)\,HC.\\ &= P\,.\,HC + Q\,.\,HC\\ &= P\,(HA+AC) + Q\,(HB-BC)\\ &= P\,.\,HA + Q\,.\,HB + P\,.\,AC - Q\,.\,BC\\ &= P\,.\,HA + Q\,.\,HB \end{aligned}$$

Since $$P\,.\,AC = Q\,.\,BC.$$

Thus the moment of R is equal to the moment of P together with the moment of Q.

If the point about which the moments are taken between the lines of action of the forces is H_1 then the two moments have opposite signs.

Also the moment of R

$$\begin{aligned} &= R\,.\,H_1C = P\,.\,H_1C + Q\,.\,H_1C\\ &= P\,(H_1A - AC) + Q\,(BC - H_1B)\\ &= P\,.\,H_1A - Q\,.\,H_1B - P\,.\,AC + Q\,.\,BC\\ &= P\,.\,H_1A - Q\,.\,H_1B. \end{aligned}$$

The proofs which have just been given can be extended to the case of three or more forces.

Thus we obtain the result that the sum of the moments of a system of forces in one plane about any point in that plane is equal to the moment of their resultant about that point.

22. Resultant of a number of Parallel Forces. We may notice (1) that this theorem enables us to find the line of action of the resultant of a number of parallel forces in a plane.

For let P_1, P_2 ... be the forces, R their resultant. Let O be any given point in the plane and let OA_1A_2 ... cut the lines of action of the forces at right angles in A_1, A_2,..., and of the resultant in G. Then we wish to determine G.

Now we have

$$R = P_1 + P_2 + P_3 + \dots.$$

Also taking moments about O

$$R \, . \, OG = P_1 \, . \, OA_1 + P_2 \, . \, OA_2 + \dots$$

Hence $$OG = \frac{P_1 \, . \, OA_1 + P_2 \, . \, OA_2 + \dots}{P_1 + P_2 + \dots}.$$

Thus (i) The position of G is found; while (ii) The theorem includes that proved in Experiment 3, viz. that the algebraic sum of the moments of a system of forces about a point in the line of action of their resultant is zero. For, if O be in the line of action of the resultant, then OG is zero and the sum of the moments vanishes.

Examples. (1) *Find the resultant of two parallel forces of* 10 *and* 12 *kilos weight acting at two points A and B* 50 *centimetres apart.*

Let the line of action of the resultant R cut AB in C.

Then $$R = 10 + 12 = 22 \text{ kilos weight.}$$

Also taking moments about A,

$$R \, . \, AC = 10 \times 0 + 12 \times 50$$
$$= 600.$$

Thus $$AC = \frac{600}{22} = 27 \cdot \dot{2}\dot{7} \text{ cm.}$$

(2) *The line of action of the resultant of two forces of* 10 *and* 15 *kilos weight is* 20 *cm. from that of the smaller force. Find the distance between the lines of action of the two forces.*

Let AB perpendicular to the lines of action of the two forces cut the line of action of the resultant in C. Then $AC = 20$ cm. and it is required to find AB.

Take moments about A

$$(10 + 15) \, AC = 15 \, . \, AB,$$

$$AB = \frac{25}{15} \, . \, 20 = \frac{100}{3} = 33 \cdot \dot{3} \text{ cm.}$$

(3) *Two men carry a pole* 24 *feet long supporting it at each end; from the middle point of the pole a mass weighing* 3 *cwt. is suspended. The weight of the pole is* 1 *cwt., and may be supposed to act at a distance of* 8 *ft. from one end. Find the weight carried by each man.*

Let AB be the pole, P and Q the upward pressures at A and B, C the middle point and G the point 8 ft. from A, at which the weight may be supposed to act.

Thus P and Q will be the pressures on the men's shoulders at A and B respectively, and their resultant must just balance that of the 3 cwt. and 1 cwt.

Thus $$P+Q=4 \text{ cwt.}$$

Also taking moments about A

$$Q \,.\, AB = 1 \,.\, AG + 3AC.$$

Thus $$Q = \frac{1 \times 8 + 3 \times 12}{24} = \frac{2+9}{6}$$

$$= 1\tfrac{5}{6} \text{ cwt. wt.}$$

Hence $$P = 2\tfrac{1}{6} \text{ cwt. wt.}$$

(4) *Masses of* 2, 4, 8 *and* 16 *lb. are placed at a series of points in a line and at distances of* 4, 3, 2 *and* 1 *feet from the edge of a table. Find the magnitude and point of application of the resultant force.*

Let the resultant force be R lb. wt. and let it act at a distance of x feet from the edge of the table.

Then $$R = 2+4+8+16 = 30 \text{ lb. wt.}$$

Also taking moments round the edge

$$Rx = 2 \times 4 + 4 \times 3 + 8 \times 2 + 16 \times 1,$$

$$x = \frac{8+12+16+16}{30} = \frac{52}{30}$$

$$= 1\tfrac{11}{15} \text{ feet.}$$

(5) *A series of forces acting at a point are represented in direction and magnitude by the sides of a polygon taken in order. Shew that the sum of their moments about any point in the plane of the polygon is zero.*

The forces are in equilibrium; they have therefore no resultant; the moment therefore of the resultant is zero; hence the sum of the moment of the forces is zero.

(6) *Forces act at the angles A, B, C of a closed polygon, each force being represented* **completely** *by one side of the polygon, and all the forces acting in the same direction round the polygon. Prove that the sum of the moments of the forces about any point in the plane of the polygon is constant.*

Let the point O (Fig. 49) be within the polygon; the forces P, Q, R etc. are completely represented by AB, BC etc.

Thus moment of $P = 2\triangle OAB$,

moment of $Q = 2\triangle OBC$, etc.

Hence sum of moments of forces

$$= 2\triangle OAB + 2\triangle OBC + 2\triangle OCD + \dots$$

$= 2$ area polygon, and this is the same for all the positions of O within the polygon.

Now let O be outside the polygon. The moment of P is again represented by $2\triangle OAB$ and so on; but in this case some of the moments are positive, some are negative. If the sum of the areas of the triangles which give

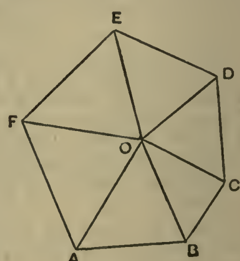


Fig. 49.

negative moments be subtracted from the sum of the areas giving positive moments, it will be found in any case that the difference is the area of the polygon.

Hence in this case also the sum of the moments is twice the area of the polygon.

23. Moment about an axis. So far we have dealt only with forces in one plane. Suppose we have any system of parallel forces and consider a line at right angles to the direction of all these forces. Each of these forces is said to have a moment round this axis; this is found by drawing a line perpendicular both to the direction of the force and to the axis, and finding the product of the length of this line and the force. This common perpendicular is the shortest distance between the direction of the force and the axis; hence the moment of a force about an axis, perpendicular to the direction of the force, is the product of the force and the shortest distance between the axis and the direction of the force. We may put this otherwise thus. Pass a plane perpendicular to the axis through the direction of the force, then the moment of the force round the axis is the same as its moment round the point in which the axis cuts this plane.

If the forces be not all at right angles to the axis each force can be resolved into two components, one at right angles to the axis the other in a plane through the axis. The product of the component at right angles to the axis into the shortest distance from the axis of its line of action is, in this case, called the moment of the force about the axis.

PROPOSITION 16. *To prove that for any system of parallel forces the sum of the moments of the forces about an axis perpendicular to their directions is equal to the moment of the resultant about the same axis.*

Let two parallel forces, P, Q be impressed on two particles A, B, Fig. 50, in directions perpendicular to the plane of the paper. Let Ox be a line in the plane of the paper about which moments are required, and let AL and BM be perpendicular on Ox.

Join AB and divide it in C so that

$$P \cdot AC = Q \cdot BC.$$

The resultant $(P+Q)$ of P and Q acts at C.

Draw CN perpendicular to Ox and HCK parallel to Ox, to meet AL and MB (produced) in H and K respectively.

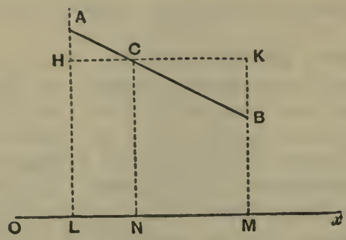


Fig. 50.

Then $$\frac{P}{Q} = \frac{BC}{AC} = \frac{BK}{AH}.$$

Therefore $$P \,.\, AH = Q \,.\, BK.$$

Now the moment of the resultant

$$= R \,.\, CN = (P + Q)\, CN = P \,.\, HL + Q \,.\, KM$$
$$= P(AL - AH) + Q(BM + BK)$$
$$= P \,.\, AL + Q \,.\, BM - P \,.\, AH + Q \,.\, BK$$
$$= P \,.\, AL + Q \,.\, BM$$

= Sum of moments of P and Q.

In the same way the proposition can be proved for any number of parallel forces.

The proof may also be extended in a similar way to any system of forces; for our purposes it is sufficient to have established it for a system of parallel forces and to have shewn that the sum of the moments of any system of parallel forces, about an axis perpendicular to the direction of the forces, is equal to the moment of their resultant about the same axis.

For examples of the use of this Proposition, see Section 38.

EXAMPLES.

PARALLEL FORCES.

1. Four equal and like parallel forces act at the angular points of a quadrilateral $ABCD$. E is the middle point of AB and F of CD. Prove that the centre of the forces is the middle point of EF. Deduce that the lines joining the middle points of the opposite sides of a quadrilateral bisect one another.

2. A uniform rod $ABCD$ moveable about a fulcrum, and thirty feet in length, has weights P, $3P$, $5P$, $7P$ attached to the rod at A, B, C and D, which are at equal distances apart. If the rod be in equilibrium, find the distance of the fulcrum from A.

3. A uniform bar of length 2 ft. 8 in. and weight $5\frac{1}{4}$ lb. is supported on a smooth peg at one end and by a vertical string distant 4 inches from the other end. Find the tension of the string.

4. Two light rods AB, BC are rigidly connected at B and meet at right angles. Weights W and W' are attached at A and C. If the system can turn about B shew that the tangent of the angle which AB makes with the horizon is

$$\tan^{-1}\frac{W}{W'}\cdot\frac{AB}{BC}.$$

5. A straight uniform heavy rod of length 6 feet has weights of 15 and 22 lb. attached to its ends, and rests in equilibrium when placed across a fulcrum distant $2\frac{1}{2}$ feet from the 22 lb. weight. Find the weight of the rod.

6. A straight uniform rod of length 6 feet and weight 11 lb. is placed across a fulcrum distant $2\frac{1}{2}$ feet from one end to which a weight of 26 lb. is attached. What weight must be attached to the other end so that there may be equilibrium?

7. A uniform bar of length 3 ft. 6 in. and weight 9 lb. is supported on a smooth peg at one end and by a vertical string distant 6 inches from the other end. Find the tension of the string.

8. A straight light rod 2 feet long rests in a horizontal position between two fixed pegs, placed at a distance of 3 inches apart, one of the pegs being at one end of the rod; a weight of 5 lb. is suspended at the other end; find the pressure on each of the pegs.

9. Let P and Q represent two like parallel forces acting at the points A and B respectively of a body. Let C be the point in the straight line AB through which their resultant R acts.

If $R=14\frac{1}{2}$ lb., $Q=8$ oz., and $AB=58$ inches; find AC and BC.

10. A uniform beam weighing 10 cwt. and lying horizontally between supports 50 ft. apart carries additional weights of 3 cwt., 5 cwt. and 8 cwt., at distances of 10 ft., 20 ft., and 35 ft. respectively from one support. Find the proportion of the whole weight born by each support.

11. *ABCD* is a square; *E*, the middle point of *AB*, is joined to *C*; *BD* is joined. Forces of 4 lb. and 6 lb. act in *AB*, *BC* respectively; of 3 lb. and 2 lb. in *AD*, *DC* respectively; a force of $\sqrt{2}$ lb. in *BD*, and one of $5\sqrt{5}$ lb. in *CE*; shew, *by using moments alone*, that the system is in equilibrium.

12. A rod *AB* moveable about a hinge *A*, has a weight of 20 lb. hung on to *B*; *B* is tied by a string to a point *C* vertically above *A* and such that *CB* is 6 times *AC*: find the tension in the string *BC*.

13. A dog-cart loaded with 4 cwt. exerts a pressure on the horse's back of 10 lbs.; find the position of the centre of gravity of the load, the distance between the pad and axle being 6 feet.

14. Two forces act on a body which can move round a fixed point and the body remains at rest. Shew that the moments of the forces round the point are equal.

15. What is meant by the moment of a force about a point?

A man and a boy carry a weight of 55 pounds between them by means of a pole $5\frac{1}{2}$ feet long, weighing 20 pounds. Where must the weight be placed so that the man may bear twice as much of the whole weight as the boy?

16. Find the magnitude and position of the resultant of four forces *P*, 2*P*, 3*P*, and 4*P* acting along the sides of a square taken in order.

17. The sides of a square are 15 inches long, at the ends of one side are two weights of 3 lb. each, at the ends of the opposite side two weights of 5 lb. each; where does the resultant of the four weights act?

18. A thin board in the form of an equilateral triangle, and weighing 1 lb. has one quarter of its base resting on the end of a horizontal table, and is kept from falling over by a string attached to its vertex and to a point on the table in the same vertical plane as the triangle. If the length of the string be double the height of the vertex of the triangle above the base, find its tension.

*CHAPTER III.

COUPLES.

24. Theorems about Couples.

Definitions. (i) *Two equal and opposite parallel forces constitute a* Couple.

(ii) *The line drawn at right angles to the directions of the two forces is called the* Arm *of the couple.*

(iii) *The product of either force into the perpendicular distance between the lines of action of the two forces—the arm of the couple—is called the* Moment of the Couple.

(iv) *A line drawn at right angles to the plane containing the two forces and proportional to the moment of the couple is called the* Axis *of the couple.*

(v) *The couple is said to be positive when, to an observer looking along the axis from the point of application of one of the forces, the forces tend to turn the body on which they are impressed in the same direction as the hands of a clock appear to move.*

We are now prepared for some propositions about couples.

Proposition 17. *The algebraic sum of the moments of the forces which constitute a couple about any point in the plane of the couple is constant and is equal to the moment of the couple.*

Let each force of the couple be P and let O, Fig. 51, be any point in the plane of the couple. Draw OAB perpendicular to the lines of action of the two forces.

Then the algebraic sum of the moments of the forces is

$$P \cdot OB - P \cdot OA.$$

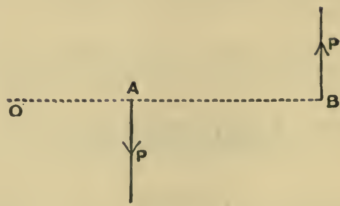


Fig. 51.

Thus the sum of the moments

$$= P(OB - OA) = P \cdot AB$$
$$= \text{moment of couple.}$$

PROPOSITION 18. *Two couples impressed on a rigid body in one plane balance if their moments are equal and opposite.*

Let the forces of the one couple be P, P and its arm p, the forces of the other couple Q, Q and its arm q. Then we have that the moment $P \cdot p$ is equal to the moment $Q \cdot q$.

(i) *If the lines of action of P and Q meet.*

Let two of the forces P, Q meet in O, Fig. 52, and the other two P, Q in O'. Draw $O'M$, $O'N$ perpendicular to the directions of P and Q respectively, then $O'M = p$ and $O'N = q$.

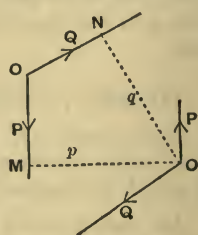


Fig. 52.

Thus we have the result that

$$P \cdot O'M = Q \cdot O'N,$$

or the moment of P about O is equal and opposite to that of Q about O. Hence by Section 22, O' is on the line of action of the resultant of P and Q which act at O.

Similarly, by drawing perpendiculars from O on the lines of action of the two forces P, Q which act at O', we can prove that O is on the line of action of the resultant of these two forces. These two resultants are equal in amount and the one acts at O' along $O'O$ the other at O along OO'. Hence they balance each other. Thus the two couples are in equilibrium.

(ii) *If the lines of action of the two forces do not meet but are parallel.*

Let $ACBD$ perpendicular to the common direction meet them as shewn in Fig. 53. We know that $P \cdot AB = Q \cdot CD$.

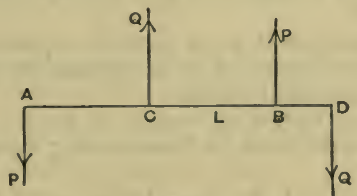


Fig. 53.

Then the resultant of P at B and Q at C is $P+Q$ acting upwards at a point L where

$$\begin{aligned}(P+Q)\,AL &= Q \cdot AC + P \cdot AB\\ &= Q \cdot AC + Q \cdot CD\\ &= Q\,(AC+CD) = Q \cdot AD.\end{aligned}$$

Again the resultant of P at A and Q at D is $(P+Q)$ acting downwards at L' where

$$\begin{aligned}(P+Q)\,AL' &= P \cdot 0 + Q \cdot AD\\ &= Q \cdot AD.\end{aligned}$$

Thus $$(P+Q)\,AL = (P+Q)\,AL',$$

or $$AL = AL'.$$

Hence L and L' coincide, thus we have $P+Q$ acting upwards and $P+Q$ acting downwards at the same point.

Hence the system is in equilibrium.

It follows therefore from this proposition that we may alter the direction of the forces of a couple, keeping the two parallel, without modifying its effect. We may also alter the forces, so long as we alter in the inverse ratio the distance between them and thus keep the moment constant.

Further we may alter the points at which we consider the forces impressed, if only the moment remains unchanged.

Thus the forces of a couple may have any value, and one of them may be supposed to be impressed at any point in the plane of the couple, so long as the moment of the couple remains unchanged in magnitude and direction.

PROPOSITION 19. *Any system of couples impressed on a rigid body in one plane is equivalent to a single resultant couple whose moment is the algebraic sum of the moments of the individual couples.*

Since two couples of equal moment, opposite in direction, balance, any two couples of equal moment, the same in direction, produce equivalent effects.

Thus we may replace any couple by another couple of the same moment with one of its forces passing through any given point.

Suppose now that P_1, P_2 ... be the forces of the various couples and p_1, p_2 ... their arms.

Replace the couples by another set of couples having the same forces P_1, P_2... and the same arms but such that all the forces are parallel, and that one force of each couple passes through a fixed point O.

Let OA_1A_2, ... Fig. 54, be perpendicular on the lines of actions of the forces and meet the second force of each couple respectively in A_1, A_2....

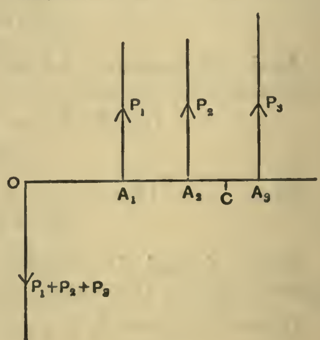


Fig. 54.

Then we have a system of parallel forces P_1, P_2 ... etc. at O, together with a second system of forces in directions opposite to these, viz. P_1 at A_1, P_2 at A_2 etc.

Moreover

$$OA_1 = p_1,\ OA_2 = p_2 \text{ etc.}$$

Now the first system of forces has a resultant R acting at O. Also $R = P_1 + P_2 + ...$

The second system has a resultant also equal to R at some point C in OA_1A_2... the direction of this force R being opposite to that of R at O.

The position of C is found by taking moments about O and is given by

$$\begin{aligned} R \cdot OC &= P_1 \cdot OA_1 + P_2 \cdot OA_2 + \dots \\ &= P_1 \cdot p_1 + P_2 \cdot p_2 + \dots \\ &= \text{sum of moments of original forces.} \end{aligned}$$

Now R at O and R at C, acting in opposite directions form a couple whose moment is $R \cdot OC$, this we have just seen is the sum of the moments of the original couples.

Thus any system of couples in a plane is equivalent to a resultant couple, whose moment is the sum of the moments—each with its proper sign—of the original couples.

PROPOSITION 20. *A force and a couple impressed on a rigid body cannot maintain it in equilibrium.*

Let O, Fig. 55, be any point in the line of action of the force P. Change the arm of the couple, keeping its moment constant, until the forces of the couple are also P. Place the couple so that one of its forces acts at O in a direction opposite to the impressed force P. Then we have two equal and opposite forces P impressed at O and a third force P in a direction parallel to these but at a distance p from them. The two forces at O balance, and we are thus left with an unbalanced force acting at A parallel to the impressed force P but at a distance p from it. Such a force cannot maintain equilibrium.

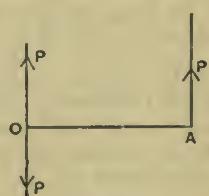


Fig. 55.

PROPOSITION 21. *A force impressed on a rigid body at any point is equivalent to an equal and parallel force, impressed at any other point, together with a couple whose moment is the moment of the force about that point.*

Let a force P be impressed on a body at any point A, Fig. 56. Let O be any other point of the body and draw ON perpendicular to the direction of P. At O introduce two equal and opposite forces P_1, P_2 equal and parallel to P. This will not affect the equilibrium. Then P at A and P_2 at O constitute a couple whose moment is $P \cdot ON$ and there is left a single force P_1 at O, equal and parallel to P.

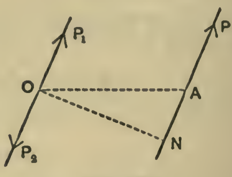


Fig. 56.

Thus the force P at A is equivalent to an equal and parallel force P at O together with a couple of moment $P \cdot ON$ about O.

PROPOSITION 22. *Any system of forces acting at different points of a rigid body in one plane is equivalent to a single resultant force acting at any given point of the plane together with a couple, whose moment is equal to the sum of the moments of the forces about that point.*

For, let there be a number of forces P, Q, R... acting at points A, B, C. Let O be any given point in the plane. Then by the last Proposition each of the given forces is equivalent to an equal and parallel force through O together with a couple whose moment is the moment of the force about O.

The forces P, Q, R... impressed at O have in general a single resultant. The couples have in general a single resultant couple whose moment is equal to the sum of the moments of the couples, and therefore to the sum of the moments of the forces about O.

PROPOSITION 23. *To find the conditions of equilibrium of a system of forces in one plane impressed on a rigid body.*

Such a system is, we have seen, equivalent to a single resultant force and a single resultant couple. Since a single force cannot balance a couple it is necessary and sufficient for equilibrium (i) that the resultant force should be zero, (ii) that the resultant couple should be zero.

In order that the resultant force may be zero the sum of its components in any two directions at right angles must be zero,

Since the moment of the resultant couple about any point is the sum of the moments of the forces about that point, it is necessary in order that the resultant couple may vanish that the sum of the moments of the forces about any point in the plane should vanish.

If the resultant force is zero we are left with a resultant couple; now the moment of a couple is constant, hence in this case the sum of the moments of the forces is constant about any point in the plane; and therefore if the sum of the moments of the forces vanishes about one point, it will vanish about all.

The necessary and sufficient conditions for equilibrium of a system of forces in one plane therefore are

(i) *The sum of the resolved parts of the forces in each of two directions at right angles is zero.*

(ii) *The sum of the moments of the forces about some one point in the plane is zero.*

Conditions equivalent to these hold for the case of forces not acting in one plane.

The following **Examples** illustrate the foregoing propositions.

(1) *Forces of* 4, 5, 6, 7 *lb. weight act at the angular points* A, B, C, D *of a square lamina each side of which is* 1 *ft. parallel to the sides* AB, BC, CD, DA *respectively. Find the resultant force and resultant couple about* A.

For the resultant force. The 4 and 6 lb. wt. are equivalent to a force of 2 lb. wt. parallel to BA, while the 5 and the 7 lb. are equivalent to a force of 2 lb. wt. parallel to DA. Hence the resultant force is $2\sqrt{2}$ lb. wt. in a direction bisecting the angle DAB and acting towards A.

For the resultant couple take moments about A, the moments of the 4 and the 7 lb. weight are zero, for these forces pass through A.

Hence the moment about A is

$$5 \cdot AB + 6 \cdot AD,$$

and since AB and AD are each 1 ft. the moment of the couple is 11 ft.-lb.

It may be noticed that the moment of a couple may be measured like work in ft.-lb., or more generally, that it is found by determining the product of a force and a distance. This will be found to be important.

(2) *Six forces impressed on a rigid body are represented completely by the sides of two triangles ABC, DEF; the forces act in opposite directions round the triangles; find the condition that the system may be in equilibrium.*

The resultant force is found by supposing all the forces to be impressed at a point unchanged in magnitude and direction. It is clearly zero, for forces acting at a point represented in direction and magnitude by the sides of the triangle ABC (Fig. 57) are in equilibrium, as are also those

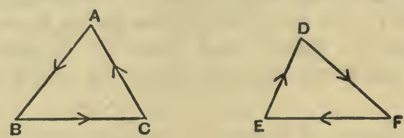


Fig. 57.

represented by the sides of DEF.

Thus the resultant is a couple.

Now the moment of forces represented completely by the sides of a triangle is twice the area of the triangle.

Thus the moment of the resultant couple is twice the difference between the areas of the two triangles. In order therefore that the forces may be in equilibrium, it is necessary that the areas of the two triangles should be equal.

(3) *The three angular points A, B, C of a triangle are each joined to two points P, Q in the plane of the triangle.*

Forces represented completely by PA, PB, and PC, AQ, BQ, and CQ act on the triangle. Shew that the resultant is a force parallel and equal to $3PQ$.

The resultant of PA, AQ (Fig. 58) acting at A is a force at A equal and parallel to PQ, for three forces at A represented by PA, AQ and QP will be in equilibrium.

Similarly the resultant of PB, BQ is PQ at B, while of PC, CQ it is PQ at C.

We have therefore three equal forces each equal to PQ acting at the angular points A, B, C; the resultant of these is a force parallel to PQ and represented by $3PQ$.

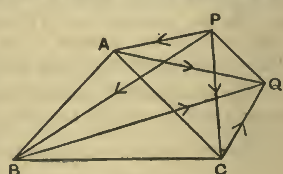


Fig. 58.

CHAPTER IV.

WORK. EQUILIBRIUM.

25. Work done by a Force. A definition of Work has been given (Dynamics, § 105), and it has been seen that work is done when the point of application of a force is moved in the direction of the force. Thus if a point A, Fig. 59, is displaced to A', and $A'A_1$ be drawn at right angles to AB, the line of action of a force F impressed at A, then work is done by the force, and the work done is measured by the product $F.AA_1$.

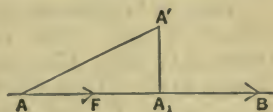


Fig. 59.

This quantity measures the product of either (i) the force and the displacement of the point of application resolved in the direction of the force, or (ii) the product of the displacement and the component of the force resolved in the direction of the displacement.

Moreover we have seen (Dynamics, § 103) that work done is one of the meanings which may be given to the term Action in Newton's statement of the third law of Motion.

26. Projection. In Figure 59, the line AA_1 is often spoken of as the projection of AA' in the direction of the force.

In general, if AB, Fig. 60, be any line and CD a second line, and if further AC and BD be perpendicular from A and B respectively on CD, then CD is the orthogonal projection, or more simply the projection of AB on the second line. If AB represents a force then CD represents the component of the force in the direction CD, it is sometimes spoken of as the projection of the force in that direction.

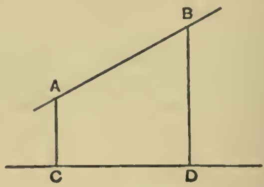


Fig. 60.

In finding, then, the work done by a force, we project the force in the direction of displacement and then multiply together the displacement and the projection of the force. Now let a number of forces P_1, P_2, P_3 represented in direction and magnitude by the lines OA, AB, BC, Fig. 61, etc. be impressed at a point; let Ox be any direction through the point; consider first the case of three forces represented by OA, AB, BC and join OC.

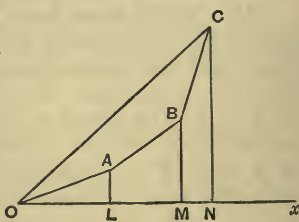


Fig. 61.

Then OC represents the resultant of the three forces. Draw AL, BM, CN perpendicular to Ox. Then AL is the projection of P_1, LM the projection of P_2 and MN the projection of P_3. Also ON is the projection of R, the resultant.

And since $ON = OL + LM + MN$, we see that the projection of the resultant is the sum of the projections of the components, or, in other words, the resolved part of the resultant is the sum of the resolved parts of the components. (See Prop. 11.)

27. Work done by a system of forces.

Proposition 24. *If a system of forces be impressed on a particle and the particle receive any displacement the work done by the forces is equal to the work done by the resultant.*

Consider the case of three forces P_1, P_2, P_3 represented by OA, AB, BC, Fig. 62. Let Ox be the direction of the displacement and let a be its amount.

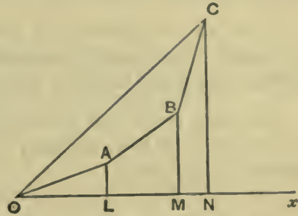


Fig. 62.

Draw AL, BM, CN perpendicular to Ox, then OL, LM, MN represent respectively the components of the forces in the direction Ox and ON represents the component of the resultant.

Hence the work done by the forces

$$= OL \,.\, a + LM \,.\, a + MN \,.\, a$$
$$= (OL + LM + MN)\, a$$
$$= ON \,.\, a$$

= the work done by the resultant.

If one of the forces act in a direction such as AB, Fig. 63, the result is still the same, for the work done by P_2 is in this case negative since the displacement takes place in a direction opposite to the force and M is to the left of L.

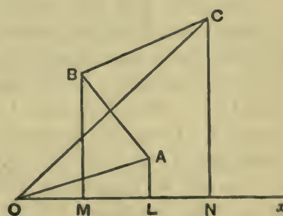


Fig. 63.

Thus the work done by forces

$= (OL - LM + MN)\, a = ON \,.\, a$

= the work done by the resultant.

If the forces form a system in equilibrium, then the point in the diagram, corresponding to C, which forms one extremity of the line representing the last force, coincides with O, the sum of the components of the forces in any direction is zero, and the work done is zero.

Thus, if a system of forces impressed on a particle be in equilibrium, no work is done in any displacement of the particle, provided that the forces are not altered by the displacement.

If we make the displacement very small, we may, even in cases in which the forces do depend on the position of the

particle, assume without error in calculating the work that the forces remain unchanged.

Thus the conditions of equilibrium, for a system of forces impressed on a particle, express the fact that the particle can be slightly displaced without expenditure of energy.

Starting from this proposition as an axiom we could establish the various laws already obtained as to the composition and resolution of forces.

***28. Work done on a rigid body.** We proceed now to shew that the conditions of equilibrium for a *body* under a system of forces in one plane also express the fact that if the body be in equilibrium no expenditure of energy is necessary to give it a slight displacement.

PROPOSITION 25. *To find the work done by a force when a body on which it is impressed receives a slight rotation about an axis at right angles to the direction of the force.*

Let AB, Fig. 64, be the direction of the force P impressed in the plane of the paper on a body. Let the body be turned about the point O through a small angle θ so that OA is brought into the position OA'; then AA' is the displacement of A and if the angle be very small we may treat AA' either as a small straight line or as an arc of a circle.

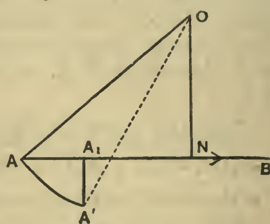


Fig. 64.

Moreover AA' is ultimately, when the angle θ is very small, at right angles to OA.

Draw $A'A_1$ perpendicular to AB the direction of the force, and ON perpendicular from O on the same line AB.

Then since OAA' is a right angle

$$A_1AA' = 90 - OAN = NOA,$$

and the angles at A_1 and N are both right angles, thus the triangles NOA, A_1AA' are similar.

Hence $$\frac{A_1A}{AA'} = \frac{ON}{OA}.$$

Thus $$A_1A = ON \cdot \frac{AA'}{OA}.$$

Now $\frac{AA'}{OA}$ is the circular measure of the angle θ.

Hence $$A_1A = ON \times \theta.$$

Now the work done by the force P is equal to $P \cdot AA_1$.

Thus

Work done $= P \cdot AA_1 = P \cdot ON \times \theta = \theta \times$ moment of P about O.

Hence, when a body is displaced through an angle θ about an axis at right angles to the impressed force, the work done by the force is found by multiplying the moment of the force about the axis by the angular displacement.

Now the circular measure of an angle is merely a number, the ratio of two lines, we see therefore how it is that the moment of a force and work are both measured in the same units, foot-pounds, foot-poundals or centimetre-dynes as the case may be.

PROPOSITION 26. *To find the work done, by a system of forces impressed on a body in one plane, when the body receives a small rotation in that plane about some point.*

If a number of forces are impressed on the body then θ, the angle through which the body is rotated, is the same for them all.

Hence, if O be the point about which rotation takes place, then the work done by each force is found by multiplying its moment about O by the angle θ.

Hence the whole work done

$$= \theta \,\{\text{sum of moments of forces about } O\}.$$

Hence if the sum of the moments about any point is zero no work is done during any small rotation about that point.

***29. Equilibrium of a rigid body.** Now we have seen that one of the conditions of equilibrium of a body under a system of forces in one plane is, that the sum of the moments of the forces about any point in the plane should be zero. If this condition be satisfied no work is done by turning the body through a small angle about any point in the plane.

Again any system of forces is equivalent to a resultant force acting at O and a resultant couple about O. The moment of this couple is the sum of the moments of the forces.

If O remains fixed, work is done only by the resultant couple, none is done by the resultant force, and the work done by the couple is found by multiplying its moment by the circular measure of the angle turned through.

Again if all points on the body are displaced the same amount, no work will be done by the resultant couple. For the work done by one force of the couple is equal to that done against the other.

Now any motion of a rigid body in one plane is made up of a translation, in which all points of the body receive equal parallel displacements, and by which some one point of the body is brought to its new position, together with a rotation of the body as a whole about this new position.

For the position of the body will be known if we know the position of two points; suppose then that the two points A, B

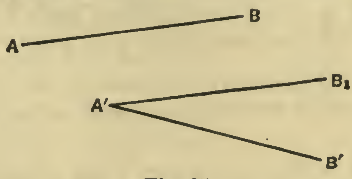


Fig. 65.

(Fig. 65) come to the position A', B', so that $A'B'$ is equal to AB. We can bring AB to the position $A'B'$ in two steps.

First keep it parallel to itself and move A to A'. Then AB will come in the position $A'B_1$ parallel to AB. Secondly turn $A'B_1$ about A' till B_1 comes to B', then AB has been brought to $A'B'$.

If a system of forces be impressed on the body the system is equivalent to a resultant R acting at A and a couple about A; let G be the moment of the couple.

Owing to the first displacement the couple does no work, the whole work done is the product of R into the component of the displacement AA' in the direction of R.

Owing to the second displacement since A' remains fixed the force R does no work. All the work done is done by the couple G and is equal to the product of G and the angle $B_1A'B'$.

Thus, if ϵ denote the angle between the direction of R and AA', and θ the angle $B_1A'B'$, the whole work done is given by

$$R \,.\, AA' \cos \epsilon + G \times \theta.$$

In order then that no work should be done in any small displacement by which the body is brought from one position to another it is necessary and sufficient that both the resultant force and the resultant couple should vanish.

Now the general form of the condition of equilibrium is that both the resultant force and the resultant couple should be zero. Thus in this case the condition of equilibrium expresses the fact that when a body is in equilibrium no expenditure of energy is needed to produce a small displacement.

Work is done by some forces and against others, if there be equilibrium these two amounts of work are equal; in Newton's phrase the action is equal and opposite to the reaction.

30. Virtual Velocities. The principle which we have just proved for the case of forces in one plane is often spoken of as the Principle of Virtual Velocities.

It may be enunciated thus:

Suppose each point of a rigid body to which a system of forces is applied to receive any small displacement consistent with its geometrical relations[1]. *Multiply each force of the*

[1] By this phrase we mean that the displacements of each point must be such as can take place without altering the form or size of the body.

system by the component of the displacement of its point of application estimated in the direction of the force; then form the sum of the products so obtained. If the forces be in equilibrium this sum is zero, and conversely, if the sum be zero for all possible displacements, the system of forces is in equilibrium.

We may state this otherwise thus:

Calculate the total work done in any small possible displacement.

Then (i) if the forces are in equilibrium this work is zero, (ii) if the work is zero *for all such displacements* then the forces are in equilibrium.

The word Virtual is used in describing the Principle because the displacements considered are not actually made; they are hypothetical and the work is calculated *on the supposition* that they are made. The displacements are spoken of as velocities because it is usual to suppose them to take place in the same time. The Principle is also called the Principle of Virtual Work.

A number of problems can most conveniently be solved by the direct application of this Principle. Examples are given in Section 33. For the present we are concerned with shewing that the conditions of equilibrium merely express the fact, *that no expenditure of energy is required in order to displace slightly a body under a system of impressed forces in equilibrium.*

31. Conditions of Equilibrium. It has been shewn that a system of forces impressed in one plane on a rigid body is equivalent to a single force and a couple, and that for equilibrium both the force and the couple must be zero. It has also been pointed out that this condition expresses the fact that no expenditure of energy is needed to give the body a slight displacement.

Now these conditions can be put into various forms each of which may be useful in special cases. We proceed then to

illustrate them by some problems and to state various useful theorems.

In solving problems the two following rules will express the conditions of equilibrium in the most straightforward manner.

(i) *Equate to zero the sum of the components of the forces in two convenient directions at right angles.*

(ii) *Equate to zero the sum of the moments of the forces about some convenient point.*

In this way three equations are obtained and the problem, if determinate, can be solved.

Examples. (1) *A uniform ladder of known length l and weight W rests on rough ground against a smooth vertical wall at a known angle α with the horizon. Find the reaction at the points of contact of the wall and the ground, assuming the weight of the ladder to be equivalent to a single force W acting at its middle point.*

Let AB (Fig. 66) be the ladder, BC the wall. Let R be the reaction at the wall. Since the wall is smooth, the direction of R is horizontal. Let P be the reaction at the ground and let it make an angle θ with the horizon.

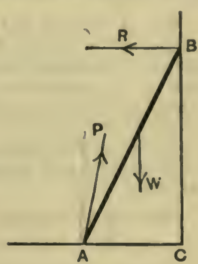


Fig. 66.

Since R acts horizontally and W vertically, the horizontal and vertical directions will be two "convenient" directions at right angles in which to resolve.

The vertical components of the forces are W downwards, and $P \sin\theta$ upwards. The horizontal components are R outwards from the wall and $P\cos\theta$ to the wall.

Thus:

Resolving vertically,

$$W = P\sin\theta \qquad\qquad (1).$$

Resolving horizontally,

$$R = P\cos\theta \qquad\qquad (2).$$

For the third condition we may take moments about *any* point, but since the direction of P passes through A, by taking them about A we eliminate two of our unknowns, P and θ; it will clearly be "convenient" to do this.

Take moments about A,

$$R\,.\,BC = W\,.\,\frac{AC}{2} \qquad\qquad (3).$$

But by geometry,

$$BC = AB \sin \alpha = l \sin \alpha,$$

$$AC = AB \cos \alpha = l \cos \alpha.$$

Hence from (3),

$$R = \tfrac{1}{2} W \cot \alpha.$$

From (1) and (2),

$$P^2 = W^2 + R^2$$

$$= W^2 \left(1 + \frac{\cot^2 \alpha}{4}\right)$$

$$P = W \sqrt{(1 + \tfrac{1}{4} \cot^2 \alpha)},$$

and

$$\tan \theta = \frac{W}{R} = 2 \tan \alpha.$$

Hence R, P and θ are determined.

We might have solved the problem by resolving in two other directions or taking moments about some other point, but less conveniently. For a different solution see page 78.

(2) *A uniform rod AB of length l and weight W is hinged at A to a vertical wall; the end B is connected by a horizontal string to the wall, and the rod is inclined to the wall at an angle α; a weight W′ is suspended from B. Determine the tension in the string and the direction and magnitude of the pressure on the hinge.*

Let T be the tension in the string. X and Y, in the directions indicated, the horizontal and vertical components of the force at the hinge.

Resolving vertically,

$$Y = W + W'.$$

Resolving horizontally,

$$T = X.$$

Take moments about A,

$$T l \cos \alpha = W' l \sin \alpha + W \frac{l \sin \alpha}{2},$$

$$T = (W' + \tfrac{1}{2} W) \tan \alpha.$$

Thus the forces are all determined.

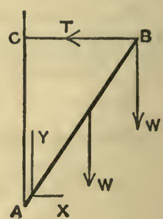


Fig. 67.

32. Problems on Forces in one plane. The conditions of equilibrium can be put into different forms. Thus:

PROPOSITION 27. *To prove that a system of forces is in equilibrium if the sum of the moments of the forces about each of three points not in the same straight line is zero.*

For let the sum of the moments about A be zero, then the system is either in equilibrium or has a resultant force through A.

If the sum of the moments about B is also zero, this resultant must pass through B. Thus it must act in the line AB. But similarly since the sum of the moments about C is also zero the resultant acts along AC; and since A, B, C are not in a straight line this is impossible, hence the resultant is zero, and the system is in equilibrium.

In the case in which the forces acting on the body reduce to three, the following proposition is useful.

PROPOSITION 28. *When three forces in one plane maintain a body in equilibrium their lines of action meet in a point or are parallel*[1].

(i) Let us suppose the directions of two of the forces meet in a point O.

Replace these two by their resultant acting through O. Then the resultant and the third force maintain equilibrium, hence the line of action of the third force passes through O.

(ii) Suppose that the directions of two of the forces are parallel, replace these by their resultant which will also be parallel to the two forces.

Then this resultant must balance the third force. Hence the three forces act in parallel directions.

Corollary. By combining this with the triangle of forces we see that, if three forces whose directions are not parallel maintain a body in equilibrium, they can be represented by the sides of a triangle drawn parallel to their lines of action respectively.

[1] It may be shewn that if three forces maintain a body in equilibrium their lines of action must be in one plane.

In applying this proposition to problems it frequently happens that the lines of action of two of the forces are known. By determining the point of intersection of these two lines of action, the direction of the third force can be found, and then by an application of the triangle of forces the values of the forces can be obtained. We will apply this to the problem of the ladder already solved, and to some other cases.

Examples. (1) *A uniform ladder of known length l and weight W rests on rough ground against a smooth vertical wall at a known angle α with the horizon. Find the reaction at the points of contact of the wall and the ground, assuming the weight of the ladder to be equivalent to a single force W acting at its middle point.*

Let G (Fig. 68) be the middle point of the ladder. The weight W acts vertically through G, the pressure of the wall R acts horizontally through B. Let the lines of action of these two forces meet at O, then the pressure of the ground must act along AO. Let OG meet the ground in D, then the three forces W, R, and P are parallel respectively to OD, DA, and AO, and act at O, hence they are proportional to these lines.

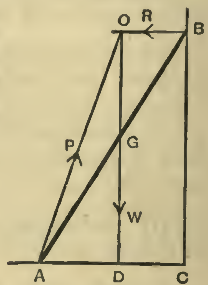


Fig. 68.

Thus $$\frac{W}{OD}=\frac{R}{DA}=\frac{P}{AO}.$$

And $$OD=BC=l\sin\alpha,$$
$$AD=\tfrac{1}{2}AC=\tfrac{1}{2}l\cos\alpha;$$

$$AO=\surd\{AD^2+DO^2\}=l\left\{\sin^2\alpha+\frac{\cos^2\alpha}{4}\right\}^{\frac{1}{2}}$$
$$=l\sin\alpha\,\{1+\tfrac{1}{4}\cot^2\alpha\}^{\frac{1}{2}}.$$

Hence $$P=W\{1+\tfrac{1}{4}\cot^2\alpha\}^{\frac{1}{2}},$$
$$R=\tfrac{1}{2}W\cot\alpha,$$

while if θ is the angle the direction of P makes with the ground,

$$\tan\theta=\frac{OD}{DA}=2\tan\alpha.$$

(2) *Two equal weightless rods AC, CB are connected by a smooth joint at C, the rod AC is free to turn about a smooth point at A, and the end B of the rod CB is free to move along a smooth groove AB. Forces P, Q in the plane ABC act at the middle points of and perpendicularly to the rods AC, CB respectively, both forces being directed towards the inside of the triangle ACB. Find the position of equilibrium and shew that if $P=2Q$ the triangle is equilateral.*

Since $CA=CB$ (Fig. 69) the triangle CAB is isosceles.

Let the angle $CBA=\theta$.

The forces on the rod CB are the pressure at B normal to AB, let this be R; the force P acting at G, the middle point of CB perpendicular to the rod, and the resistance of the hinge at C, let this be S, and let the lines of action of R and P meet in M.

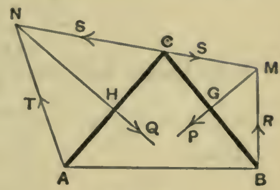


Fig. 69.

Then since there are three forces on the rod, their lines of action meet in a point. Hence S acts along CM.

Moreover $\angle GMB = GMC = \theta$,

for $\angle GBM = 90° - \theta$,

and G is the middle point of BC.

Thus MG, the direction of P, bisects the angle CMB. Hence the two forces R and S are equal, and resolving the forces at M along MG we have

$$P = 2S \cos\theta = 2R \cos\theta.$$

Now the action of BC on CA at the hinge must be equal and opposite to that of CA on BC. Hence if MC be produced to N there is a force S acting on AC at C along CN.

Let H be the middle point of AC and draw HN perpendicular to AC to meet CN in N. The force Q acts along NH.

Thus the lines of action of two of the forces on AC meet in N. The third force acting on AC is T, the pressure at the hinge A. This force must therefore act along AN, and for the rod AC we have forces S, T and Q acting at N.

Moreover $\angle HNA = \angle HNC$,

and therefore $S = T$.

Thus resolving along NH, $Q = 2S \cos HNC$.

Again in the figure

$$\angle ABC = \theta.$$

Therefore, $\angle BCM = \angle CBM = 90 - \theta$,

and $\angle ACB = 180 - 2\theta$.

Hence
$$\angle HCN = 180 - (180 - 2\theta) - (90 - \theta)$$
$$= 3\theta - 90.$$

Thus $\angle HNC = 90 - \angle HCN = 180 - 3\theta$.

Therefore,
$$Q = 2S \cos HNC = 2S \cos(180 - 3\theta)$$
$$= -2S \cos 3\theta.$$

Hence we have

$$P = 2S \cos\theta, \quad Q = -2S \cos 3\theta.$$

Therefore θ is given by the equation

$$\frac{Q}{P} = -\frac{\cos 3\theta}{\cos\theta} = 3 - 4\cos^2\theta$$
$$= 4\sin^2\theta - 1.$$

If the triangle be equilateral

$$\theta = 60^\circ, \quad \sin\theta = \frac{\sqrt{3}}{2}.$$

Hence $$\frac{Q}{P} = 3 - 1 = 2.$$
$$Q = 2P.$$

33. Virtual Velocities. A number of problems may readily be solved by the principle of work.

In applying this principle we have to suppose the system displaced and to calculate the work done in the displacement by each force. We notice (i) that if any part of the system moves over a smooth surface no work is done by the pressure of the surface, for the displacement is at right angles to the direction of the force; (ii) that if any part of the system consists of two rods connected by a smooth joint, no work is done by the mutual forces acting on the rods at the joints, for the displacement is the same for both rods at the joint, but the force which is impressed on the one rod is equal and opposite to that on the other, hence the work done is zero. The following examples illustrate the principle.

Examples. (1) *A weight P is tied to each end of a string which hangs over two smooth pegs AB in the same horizontal line. At a point O in the string midway between A and B a weight W is suspended. Find the angle which the string makes with the vertical when there is equilibrium.*

Let the angle be θ.

Let the weight W be displaced a small vertical distance x so that O (Fig. 70) may come to O', and in consequence let the weights P each rise a distance y bringing them to P'.

Then the work done by W is $W.x$, that done against each of the weights P is $P.y$.

Hence $W.x = 2P.y$.

From O draw OL perpendicular to AO'. Then since OO' is very small AL is equal to AO.

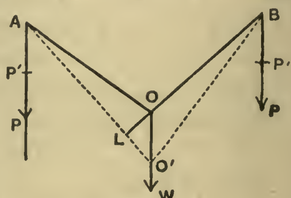


Fig. 70.

Now the length of the string from P to O is the same as that from P' to O'.

Hence

$$PP' + P'A + AO$$
$$= P'A + AL + LO'.$$

Therefore $$LO' = PP' = y.$$

And in the triangle LOO' the angle at L is a right angle, and the angle $LO'O$ is θ.

Hence $$LO' = OO' \cos\theta,$$

or $$y = x\cos\theta.$$

Therefore $$\cos\theta = \frac{y}{x} = \frac{W}{2P}.$$

This result could of course have been more readily obtained by resolving the forces at O in a vertical direction, this at once gives

$$W = 2P\cos\theta.$$

(2) *Apply the principle of work to find the position of equilibrium for the rods in Example* 2, *p.* 78.

The reactions at A, B, C all disappear from the equation of work for the reasons given above; the only forces which we have to consider are P and Q.

Suppose now that the rod AC is brought into the position AC' by being turned through a small angle about A, and in consequence let CB come to the position $C'B'$.

Let H', G' be the new positions of H and G.

And let HH', which may be treated either as a small arc of a circle about A or a short straight line perpendicular to AC, be equal to x. Then $CC' = 2x$.

Draw CL and $C'M$ perpendicular on AB, and draw $C'N$ parallel to BA to meet CL in N.

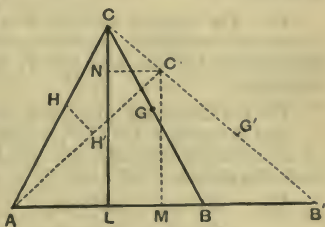


Fig. 71.

Now since G and G' are respectively the middle points of BC and $B'C'$, the component in any direction of the displacement GG' is half the sum of the components in the same direction of the displacements CC' and BB'.

Now $\angle ACC'$ is a right angle and $\angle ACB$ is $180 - 2\theta$. Hence

$$\angle C'CB = 2\theta - 90°,$$

and the projection of CC' in the direction of P is $CC' \sin C'CB$ which is equal to

$$2x \cos 2\theta.$$

Again the projection of BB' in the same direction is, since P acts inward,

$$-BB' \sin ABC.$$

Moreover $$AL = \tfrac{1}{2} AB,$$

$$AM = \tfrac{1}{2} AB'.$$

Hence $$BB' = 2LM = 2C'N$$

$$= 2CC' \sin C'CN$$

$$= 4x \sin \theta.$$

Thus the projection of BB' on the line of action of P is $-4x \sin^2 \theta$.

Hence

Projection of GG' on the line of action of P

$$= \tfrac{1}{2} \{2x \cos 2\theta - 4x \sin^2 \theta\}$$

$$= x \{1 - 4 \sin^2 \theta\}.$$

Thus the work done by Q is Qx, that done by P is

$$Px \{1 - 4 \sin^2 \theta\}.$$

Hence $$Qx + Px \{1 - 4 \sin^2 \theta\} = 0,$$

or $$\frac{Q}{P} = 4 \sin^2 \theta - 1,$$

as before.

(3) *The opposite angular points A, C, B, D of a rhombus $ABCD$ are connected respectively by two elastic strings. When the whole is in equilibrium the tension in AC is P, that in BD is Q; find the angle of the rhombus.*

Let the diagonals intersect in O (Fig. 72). Let the rhombus be displaced so that it becomes $A'B'C'D'$, so that A moves a distance AA' along OA and B a distance BB' along BO.

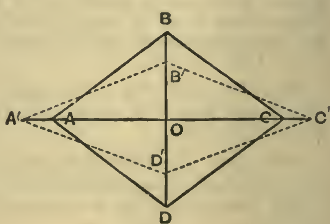


Fig. 72.

Let $OA = a$, $AA' = \alpha$,

$OB = b$, $BB' = \beta$.

Then α and β are small and the principle of work gives

$$-P \,.\, AA' + Q \,.\, BB' = 0.$$

Hence $$\frac{P}{Q} = \frac{\beta}{\alpha}.$$

Let c be the side of the rhombus, then

$$AB = c = A'B',$$

and

$$a^2+b^2=AB^2=c^2=A'B'^2=(a+\alpha)^2+(b-\beta)^2$$
$$=a^2+2a\alpha+\alpha^2+b^2-2b\beta+\beta^2.$$

And if α and β are very small we may neglect α^2 and β^2.

Hence $$2a\alpha-2b\beta=0,$$

or $$\frac{\beta}{\alpha}=\frac{a}{b}.$$

Hence $$\frac{P}{Q}=\frac{a}{b}=\tan BAC.$$

Thus the angle between the sides of the rhombus is given in terms of the tension.

EXAMPLES.

1. A uniform rod of weight W is supported from a point by two strings. One of these makes an angle of 60°, the other an angle of 30°, with the rod. Find the tensions in the strings.

2. Forces P, Q, and R act along lines OA, OB, and OC which are met by a line through O' in A, B, and C. Shew that if the resultant of the forces passes through O', then

$$P\frac{O'A}{OA}+Q\frac{O'B}{OB}+R\frac{O'C}{OC}=0.$$

3. A rod whose length is 10 feet, and which is thicker at one end than at the other, balances about its centre when 10 lb. is hung from one end and 20 from the other; while if 40 lb. instead of 20 is hung from the second end the fulcrum is at 4 feet from that end. Find the weight of the rod, and the position of its centre of gravity.

4. A heavy uniform plank 9 feet long can turn about a point 3 feet from one end. A man whose weight is double that of the plank stands upon it, with one of his feet midway between the centre of the plank and the point of support, and the other foot 2 feet from the end. Find the pressures which each of the man's feet must exert on the plank in order to preserve equilibrium.

5. A heavy rod equal in length to the radius lies in a smooth hemispherical cup, the centre of gravity of the rod being one-third of its length from one end. Shew that if θ be the angle made by the rod with the vertical $\tan\theta=3\sqrt{3}$.

6. ABC is an equilateral triangle, and forces P, P and $2P$ act along BA, AC and CB respectively. Find the magnitude of the resultant and the point at which it cuts the line AC, produced if necessary.

7. A straight rod has its ends moveable on the arc of a smooth fixed curved wire in a horizontal plane; if a string is fastened to the centre of the rod, find by a geometrical construction the direction in which it can be pulled without moving the rod.

8. Two equal heavy rods AC, BC are jointed together at C, and have their other extremities A and B jointed to fixed pegs in the same vertical line. Prove that the direction of the stress at C is horizontal, and determine, by geometrical construction, the stresses at A and B.

9. If a weight equal to the weight of either rod be attached to the centre of the lower rod, prove that if α is the inclination of each rod to the vertical, and θ the inclination to the vertical of the stress at c,

$$\tan\theta = 3\tan\alpha;$$

and that this stress is to the weight of either rod in the ratio

$$\sqrt{1+9\tan^2\alpha} : 4.$$

10. Two equal heavy uniform beams AB, BC, each of weight W, jointed at B so as to make an angle α with one another, rest in a vertical plane with the ends A, C on a smooth horizontal plane, and AC is joined by an inextensible string. Determine the tension of the string.

11. AB is a uniform rod of weight W attached by two light strings AC, BD to two points C, D in the same horizontal line. Assuming AC to be the shorter string, shew that it is possible to choose a weight P and place it on the rod so as to maintain it in a horizontal position, when P is at a distance from the middle-point of the rod equal to

$$\frac{W+P}{P}\frac{\sin(\theta-\phi)}{\sin(\theta+\phi)}\cdot\frac{AB}{2},$$

where θ and ϕ are the acute angles made with CD by AC and BD respectively.

12. A small bead P is free to move on a given smooth circular wire and is acted on by two forces represented by PA and PB, where A and B are fixed points in the plane of the ring. Find the positions of equilibrium.

13. The sides of a triangular framework are 13, 20, and 21 inches: the longest side rests on a horizontal smooth table and a weight of 63 lb. is suspended from the opposite angle. Find the tension in the side on the table.

14. A gate is hung in the usual manner by two hinges on a gate-post. Indicate the forces acting on the gate when it hangs open and in equilibrium, and shew that it may happen that the reaction of one of the hinges is wholly horizontal. Give a verbal explanation as well as a diagram.

15. A curtain-ring can slide along a horizontal pole, which is at a height of 4 feet above my hand: if a string 10 feet long is attached to the ring, shew by a diagram in what direction I must pull the string so as to move the ring with most effect, and at what point of the string I must take hold.

16. Two small heavy rings of weights W and W' connected by a light string slide on two wires in the same vertical plane making equal angles α with the horizon. If the string makes an angle θ with the horizon shew that

$$(W+W')\tan\theta=(W-W')\cot\alpha.$$

17. A ladder rests against a smooth wall, the ground being also smooth. Compare the horizontal forces which must be applied to the bottom of the ladder to preserve equilibrium, when a weight equal to the weight of the ladder is placed on the ladder at the top and bottom respectively.

CHAPTER V.

CENTRE OF GRAVITY.

34. Centre of Mass. Consider two particles A, B, Fig. 73, of mass m, m' respectively, let them be connected by an inextensible rod whose mass we may neglect, and suppose two parallel forces act on them, the force on each particle being proportional respectively to the mass of that particle. Let the two forces then be ma and $m'a$, a being some constant. These two forces have a resultant which acts at a point C in the line AB, dividing AB so that

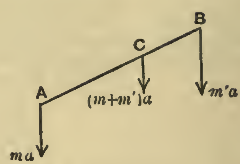


Fig. 73.

$$ma \,.\, AC = m'a \,.\, BC,$$

or
$$\frac{AC}{BC} = \frac{m'}{m}.$$

The point C is called the centre of mass of the two particles. Whatever be the position of the line AB the point C is fixed in that line and the resultant force $ma + m'a$ always acts through it.

Again if we have three masses m_1, m_2, m_3 at A_1, A_2, A_3 Fig. 74, and forces m_1a, m_2a, m_3a are impressed on these, the centre of mass of m_1 and m_2 will be a point C in A_1A_2 such that

$$m_1CA_1 = m_2CA_2.$$

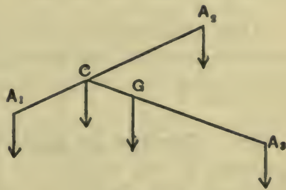


Fig. 74.

We may consider the forces m_1a and m_2a to be replaced by their resultant $(m_1 + m_2)\,a$ impressed at C. Then this force at C and m_3a at A_3 will have a resultant acting always at G a point in A_3C such that

$$(m_1 + m_2)\,CG = m_3GA_3.$$

This resultant will be $(m_1 + m_2 + m_3)\,a$. The point G is the centre of mass of the three particles.

In general, since the resultant of any number of *parallel* forces impressed on a rigid body at various points is a force equal to the sum of the individual forces, whose line of action always passes through a fixed point in the body, we see that if parallel forces m_1a, m_2a..., be impressed respectively on each of the particles m_1, m_2..., of a rigid body these forces will have a resultant equal to $(m_1 + m_2 \ldots)\,a$ whose line of action passes through a fixed point of the body. This point is called the centre of mass of the body.

DEFINITION. *The* **Centre of Mass** *of a body is the point of action of the resultant of a system of parallel forces impressed on each of the particles of the body, each force being proportional to the mass of the particle on which it is impressed. This point is fixed in the body.*

The position of the centre of mass does not depend on the direction of the parallel forces but only on their amounts and on the points at which they are impressed, thus if the body be turned in any way, the forces still remaining parallel, their resultant still acts through the same point in the body.

Again there is only one centre of mass, for if there be two, H_1 and H_2, turn the body if necessary until the line H_1H_2 is at right angles to the forces.

Then the resultant force acts in one line through H_1 and in a second parallel line through H_2, which is impossible. Hence there is only one centre of mass.

35. Centre of Gravity. The weight of a body is the resultant of the weights of the particles of which the body is composed, the weight of each particle is proportional to its mass and, if the body is small compared to the Earth, the lines joining the particles to the centre of the Earth are all parallel, so that the weights of the particles form a system of parallel forces, each being proportional to the mass of the particle on which it is impressed.

The resultant of these forces passes through a fixed point in the body—the centre of mass—or as it is more often called when considered in connexion with the weight of the body the centre of gravity.

DEFINITION. *The weights of the various particles of which a body is composed form a system of parallel forces; these forces have a resultant equal to their sum. This resultant passes through a point which is fixed in the body however it be placed. This point is called the* Centre of gravity *of the body.*

Thus we may treat the weight of a body of finite size as a single vertical force impressed on the body at a definite point; this point is its centre of gravity.

If we allow for the fact that the weights of the various particles are not strictly parallel forces, it does not follow that their resultant passes in all cases through a fixed point in the body. The body has no centre of gravity though it has a centre of mass.

It is clear that if the only impressed force be the weight of the body and the centre of gravity be supported the body will balance in any position in which it may be placed.

We shall describe first an experimental method which can sometimes be applied in order to find the centre of gravity of a body, and then shew how to determine by calculation the position of the centre of gravity for each of certain bodies.

The following proposition will be needed.

Proposition 29. *To prove that, if a body is suspended freely from one point, the centre of gravity is either vertically above or vertically below the point of suspension.*

Two forces only act on the body, viz. its weight and the force exerted at the point of support. These two forces must be equal and their lines of action must coincide.

Now the weight acts vertically through the centre of gravity. Hence the point of support must be in the same vertical as the centre of gravity, and the direction of the force at the point of support must be vertical.

Two cases of this proposition arise.

In the one, Fig. 75, the centre of gravity G is vertically below the point of support O; if the body in this position be slightly displaced it will tend to return to it; the equilibrium is said to be stable.

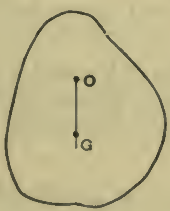


Fig. 75.

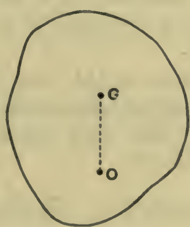


Fig. 76.

In the other case, Fig. 76, the centre of gravity G is vertically above the point of support, the body, if displaced, will not return to its original position; the equilibrium is said to be unstable.

36. Experiments on Centre of Gravity.

Experiment 4. *To find the centre of gravity of a plane lamina*[1].

Attach a string to any point A, Fig. 77, of the body and suspend it by the string.

Draw, by the aid of a plumb line hanging from the same support as the string, on one face of the lamina a vertical line AG through A. The centre of gravity lies in this line, provided the lamina be very thin.

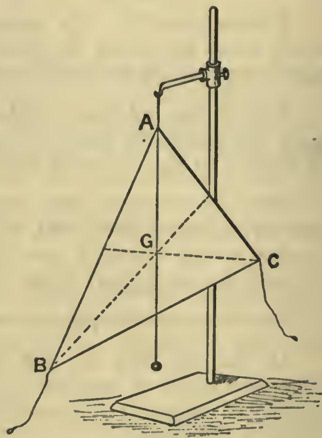


Fig. 77.

Suspend the lamina by a string attached to another point B, and draw on the same face a vertical line BG through B, intersecting AG in G. The centre of gravity lies in this vertical line. Hence the centre of gravity must be G, the point where these two lines intersect. To verify this, suspend the lamina from a third point C, the vertical line through C will also be found to pass through G.

Determine in this way the position of the centre of gravity for a triangular lamina and for a square with a corner cut off; and shew that they agree with their theoretical positions. Sections 37, 39.

Experiment 5. *To find the centre of gravity of a framework.*

On one of the bars of the frame-work a light wire is fixed, this has a ring at one end. Suspend the frame-work from any

[1] A lamina is a thin flat sheet of any material, such as a sheet of paper or cardboard or of very thin metal; for the experiment a thin wooden board will serve. The centre of gravity will be in the interior midway between the surfaces of the board.

point A, Fig. 78, and bend the wire so that a vertical string through A passes through the ring. Suspend the framework from a second point B, and keeping the string from A through the ring determine where the vertical through B cuts this string, this point will be the centre of gravity of the frame-work. Bend the wire until the ring just comes into this position, so that when suspended from A or B the vertical from the point of suspension passes through the ring. It will be found that when suspended from any other point the vertical through that point passes through the ring.

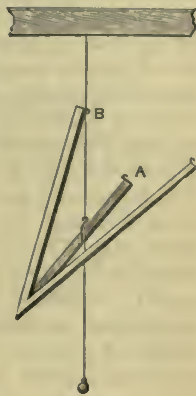


Fig. 78.

The centre of gravity of a solid body cannot be found in this manner, because of the impossibility of reaching the point in the interior of the body which is always vertically below the point of support.

37. Centre of Gravity found by Calculation. The position of the centre of gravity can be found in some cases from consideration of symmetry.

PROPOSITION 30. *To find the centre of gravity of a uniform straight rod.*

The centre of gravity is clearly the middle point of the rod, for let AB, Fig. 79, be the rod. Bisect AB in G. Let P and Q be two equal particles of the rod equidistant from G. Then the resultant of the weights of these two particles passes through G. And the whole rod can be divided into a number of such pairs of equal particles, the centre of gravity of each pair is G, hence the centre of gravity of the rod is G.

Fig. 79.

PROPOSITION 31. *To find the centre of gravity of a lamina in the form of a parallelogram.*

Let $ABCD$, Fig. 80, be a parallelogram composed of some material of uniform thickness and density.

Bisect the sides AB and CD in E and F and join EF; bisect BC and DA in H and K and join HK intersecting EF in G. Then G shall be the centre of gravity required. Divide the whole parallelogram into a series of narrow strips by lines such as PQ parallel to AB. Let PQ meet EF in R. Then since EF is parallel to BC and bisects AB and CD it also bisects PQ. Thus R is the middle point of PQ. Now we may consider the strip PQ as a uniform straight line; its centre of gravity therefore is at R. The whole parallelogram may be considered as made up of a series of strips such as PQ, the centre of gravity of each of these is on the line EF, thus the centre of gravity of the whole is on the line EF.

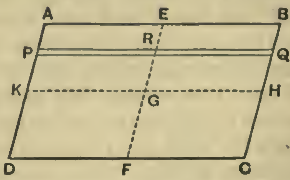


Fig. 80.

In a similar way the parallelogram may be divided up into a series of narrow strips parallel to AD; the centre of gravity of each of these will be at its middle point, that is on the line HK. Hence the centre of gravity of the whole is on the line HK.

Thus the centre of gravity of the whole parallelogram is G, the point of intersection of HK and EF.

PROPOSITION 32. *To find the centre of gravity of a uniform triangular lamina.*

Let ABC, Fig. 81, be a uniform triangular lamina.

Bisect the sides BC, CA, in D and E, and join AD and BE intersecting in G. Then G shall be the centre of gravity required. Divide the triangle into a series of narrow strips, such as PQ, by lines drawn parallel to BC. Let AD meet PQ in R.

Then since PQ and BC are parallel, and are met by the three lines AB, AD and AC we have

$$\frac{PR}{QR} = \frac{BD}{CD} = 1.$$

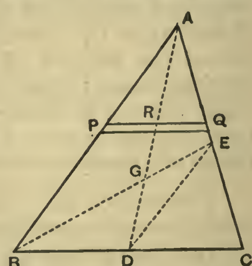


Fig. 81.

Thus R is the middle point of PQ.

Hence R is the centre of gravity of the strip PQ.

Hence the centre of gravity of all strips such as PQ lies on AD.

Thus the centre of gravity of the triangle is in AD.

Similarly, by dividing the triangle into strips parallel to CA, we can prove that the centre of gravity of the triangle is in BE.

Therefore the centre of gravity of the triangle must be G, the point where AD and BE intersect.

Again since D and E are the middle points of CB and CA respectively, DE is parallel to BA and equal to $\frac{1}{2}BA$.

Moreover, since DE and AB are parallel and are met by AD and BE,

$$\angle ADE = \angle DAB,$$

and

$$\angle BED = \angle EBA.$$

Hence the triangles GDE and GAB are similar.

Thus
$$\frac{DG}{AG} = \frac{DE}{AB} = \frac{1}{2}.$$

Hence
$$DG = \tfrac{1}{2}AG = \tfrac{1}{3}AD.$$

Similarly
$$EG = \tfrac{1}{3}BE,$$

and if CG be joined it will when produced pass through F the middle point of AB and $FG = \frac{1}{3}FC$.

Thus *the centre of gravity of a triangle is found by joining any angular point to the middle point of the opposite side, and taking a point on this line at a distance from the angle equal to $\frac{2}{3}$ds of the whole length.*

PROPOSITION 33. *To shew that the centre of gravity of a triangle is the same as that of three equal masses at its angular points.*

It is clear that the point G thus found will be the centre of gravity of three equal masses placed at the angular points of the triangle. For consider two equal mases m at B and C, their centre of gravity is D midway between them, and we may

replace them by $2m$ at D. Now since $AG = 2GD$ we have

$$m \,.\, AG = 2m \,.\, GD.$$

Thus G, which is the centre of gravity of the triangle, is also the centre of gravity of m at A and $2m$ at D. It is therefore the centre of gravity of the three masses, m at A, B and C.

PROPOSITION 34. *To find the centre of gravity of three uniform rods, BC, CA and AB forming a triangle.*

Let a, b, c be the lengths of the rods respectively, their masses are proportional to a, b and c.

Bisect the rods in D, E, F, Fig. 82, then the points D, E, F are the centres of gravity of the rods and we have to find the centre of gravity of masses a, b, c at the points D, E, F respectively. Let H, a point in FE, be the centre of gravity of masses b at E and c at F, K of masses c at F and a at D, L of masses a at D and b at E.

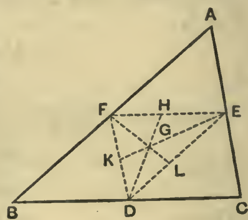


Fig. 82.

Then $\dfrac{FH}{EH} = \dfrac{b}{c} = \dfrac{AC}{AB} = \dfrac{FD}{ED}$.

Therefore DH bisects the angle FDE and we have to find the centre of gravity of $b + c$ at H and a at D. It is clearly a point in DH.

But by considering first c at F and a at D we can shew similarly that the centre of gravity required is a point in EK.

Hence the centre of gravity of the three rods is G, the point in which DH and EK coincide.

This point is clearly the centre of the circle inscribed in the triangle DEF.

38. Formulæ connected with Centre of Gravity. Formulæ can be found which enable us in many cases to obtain by calculation the position of the centre of gravity of a body.

Thus :

Proposition 35. *To find the position of the centre of gravity of a number of particles in a straight line.*

Let A_1A_2, Fig. 83, be the positions of the particles, $m_1m_2...$ their masses. Take any point O in the line and let $x_1, x_2...$ be the distances of the particles from O.

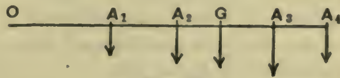


Fig. 83.

Let G be the centre of gravity.

Then G is the point of application of the resultant of a series of parallel forces proportional to m_1, m_2, etc., acting at A_1, A_2, etc., respectively.

And by Proposition 15 the moment of the resultant of these forces about O is equal to the sum of the moments of the forces. Hence, taking moments about O,

$$(m_1 + m_2 + ...)\, OG = m_1x_1 + m_2x_2 + ...$$

Hence
$$OG = \frac{m_1x_1 + m_2x_2 + ...}{m_1 + m_2 + ...}$$
$$= \frac{\Sigma(mx)}{\Sigma(m)},$$

where as before $\Sigma(mx)$ stands for the sum of a series of quantities like mx.

Proposition 36. *To find the centre of gravity of a number of particles in a plane.*

Let m_1, m_2 be the masses of the various particles placed at points A_1, A_2 etc. in a plane.

Let O, Fig. 84, be any fixed point in the plane, and Ox, Oy two lines at right angles meeting in O.

Let A_1M_1, A_2M_2, etc., be perpendicular on Ox, and A_1L_1, A_2L_2, etc., perpendicular on Oy.

Let
$$A_1L_1 = x_1,\ A_2L_2 = x_2, \text{ etc.}$$
$$A_1M_1 = y_1,\ A_2M_2 = y_2, \text{ etc.}$$

Let G be the centre of gravity and let GL, perpendicular on Oy, $=\bar{x}$, and GM, perpendicular on Ox, $=\bar{y}$.

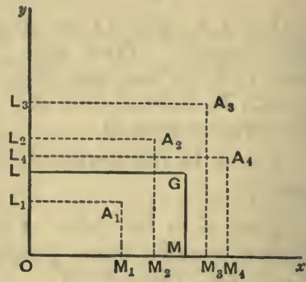


Fig. 84.

We require to find the point of application of the resultant of forces proportional to m_1, m_2, etc., impressed on particles at A_1, A_2 etc.

The point of application of the resultant is the same whatever be the direction of the forces so long only as they all remain parallel. Let us suppose then that they act perpendicularly to the plane of the paper. They are then at right angles to Ox and Oy and by Prop. 15 the moment of the resultant about Ox and Oy is equal to the sum of the moments of the forces about these lines. Hence, taking moments about Oy,

$$\{m_1 + m_2 + m_3 + \ldots\}\, GL = m_1 . A_1L_1 + m_2 . A_2L_2 + \ldots.$$

Thus $$\bar{x} = GL = \frac{m_1x_1 + m_2x_2 + \ldots}{m_1 + m_2 + \ldots} = \frac{\Sigma(mx)}{\Sigma(m)},$$

while, taking moments about Ox,

$$(m_1 + m_2 + \ldots)\, GM = m_1 . A_1M_1 + m_2 . A_2M_2 + \ldots$$

$$\bar{y} = GM = \frac{m_1y_1 + m_2y_2 + \ldots}{m_1 + m_2 + \ldots} = \frac{\Sigma(my)}{\Sigma(m)}.$$

By these two equations the position of G is determined.

A similar method can be applied to a series of particles in space. In this case we shall have to consider three axes at right angles and obtain the formulæ. We will illustrate the results by some examples.

Examples. (1) *Find the centre of gravity of masses of* 10, 20, 30, 40, *and* 50 *grammes arranged in a straight line at intervals of* 10 *cm. apart.*

Let O be the position of the 10 gramme mass. The distances of the various masses from O are therefore 0, 10, 20, 30 and 40 cm. respectively.

Let G be the centre of gravity,

then, taking moments about O,

$$OG\,(10+20+30+40+50)$$
$$=10.0+20.10+30.20+40.30+50.40,$$
$$OG=\frac{4000}{150}=26\tfrac{2}{3}\text{ cm.}$$

(2) *Masses of* 10, 20, 30 *and* 40 *grammes are placed at the four angles of a square each side of which is* 20 *cm. in length, and a mass of* 50 *grammes at the centre; find the centre of gravity of the whole.*

Let $ABCD$ (Fig. 85) be the square, E the centre, and the masses be placed as shewn in the figure.

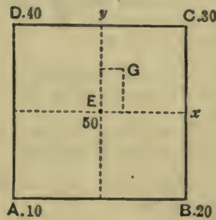


Fig. 85.

(i) *By application of the formula.*

Draw lines Ex, Ey through E parallel to the sides of the square.

Let G be the centre of gravity, x and y its distances from Ey and Ex respectively, x being measured to the right, and y upwards. The distances of the angles of the square from Ex and Ey are each 10 cm., but in taking moments about Ex it must be noted that the moments of the 30 and 40 grammes are of opposite sign to those of the 10 and 20 grammes. And similarly, when taking moments round Ey, the moments of the 10 and 40 grammes are negative, those of the 20 and 30 grammes positive.

Hence, taking moments about Ey,

$$\bar{x}\,(10+20+30+40+50)$$
$$=50.0+40.10+30.10+20.10-10.10$$
$$=500-500=0.$$

Hence $\bar{x}=0$ and G lies in Ey.

This result is obvious from the symmetry.

Taking moments about Ex,

$$\bar{y}\,(10+20+30+40+50)$$
$$=50.0+40.10+30.10-20.10-10.10$$
$$=400.$$

Therefore $$\bar{y}=\frac{400}{150}=2\tfrac{2}{3}\text{ cm.}$$

Thus the centre of gravity is on Ey at a distance of $2\tfrac{2}{3}$ cm. above E.

(ii) *By geometrical construction.*

Divide DA (Fig. 86) into 5 parts and let H be the first division from D so that

$$\frac{DH}{AH}=\frac{1}{4}.$$

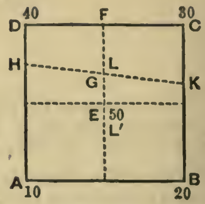


Fig. 86.

H is then the centre of gravity of 40 grammes at D and 10 grammes at A, and it is at a distance of 4 cm. from D.

Similarly, if K be a point on CB at a distance of 8 cm. from C, then K is the centre of gravity of 30 grammes at C and 20 at B.

We have now to find the centre of gravity of 50 grammes at H, 50 grammes at K and 50 grammes at E.

Join HK, cutting in L a line EF through E parallel to the side DA. Let EF meet DC in F.

Then $HL=LK$, and L is the centre of gravity of 50 grammes at each of the points H and K.

Also, since $DH=4$ cm. and $CK=8$ cm., $FL=6$ cm., and therefore $EL=4$ cm.

We have now to find the centre of gravity of 50 grammes at E and 100 grammes at L.

This will be a point G in EL such that

$$EG=2GL=\tfrac{2}{3}EL=\tfrac{2}{3}.4=2\tfrac{2}{3}\text{ cm.}$$

This is of course the same point as was found by the previous method.

(3) *Where must a mass of* 100 *grammes be placed in order that the centre of gravity of the system and the* 5 *masses described in the previous question may be at the centre of the square?*

The centre of gravity of the four particles at the angles of the square has been shewn to be at L (Fig. 86) in the line EF at a distance of 4 cm. from E. Produce LE to L' making $L'E=EL=4$ cm., and place a mass of 100 grammes at L'. The centre of gravity of 100 grammes at L and 100 grammes at L' is at E half way between them; the 50 grammes is already at E. Hence in order that the centre of gravity of the whole may be at E, the 100 grammes must be placed at L'.

(4) *A uniform wire ABC is bent at B so that the angle ABC is* 60°, *and suspended from the point A. The part AB is* a *cm. long. Find the length of BC in order that when the whole is in equilibrium BC may be horizontal.*

Let $$BC=x\text{ cm.}$$

Let H and K (Fig. 87) be the middle points of AB and BC. Draw AD vertical and HL horizontal.

Then $AB=a$, hence $BD=\frac{a}{2}$,

and $HL=\frac{a}{4}$;

also $BK=\frac{x}{2}$, hence $DK=\frac{x}{2}-\frac{a}{2}$.

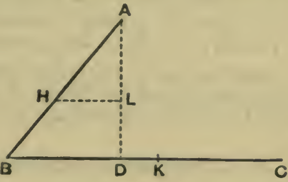


Fig. 87.

Now H is the centre of gravity of AB, and the mass of AB is proportional to its length a. Again K is the centre of gravity of BC, and the mass of BC is proportional to its length x.

Thus, taking moments about A,

$$a \cdot HL = x \cdot DK.$$

Hence $$\frac{a^2}{4}=\tfrac{1}{2}x(x-a).$$

Thus $$2x^2-2ax-a^2=0,$$

$$x=a\pm a\sqrt{(1+2)}$$

$$=a(1+\sqrt{3}),$$

since x must be positive.

(5) *Five pieces of uniform chain are hung at equidistant points on a uniform horizontal rod without weight. The shortest piece of chain is at the same distance from O, the end of the rod, as the interval between any two chains, and the lower ends of the chains lie on a straight line which passes through O; find the centre of gravity of the system.*

Let a be the distance between the chains, b the length of the shortest chain, then the lengths of the other chains are $2b$, $3b$, $4b$, $5b$, and the distances from the chains are a, $2a$,...$5a$.

Let $\bar{x}$ be the distance from O of the centre of gravity measured along the rod. Then, taking moments about O, we have

$$\bar{x}(b+2b\ldots 5b)=ab+2a\cdot 2b+\ldots 5a\cdot 5b,$$

$$\bar{x}=\frac{a(1+2^2+\ldots 5^2)}{1+2+\ldots 5}=\frac{55a}{15}$$

$$=\frac{11a}{3}.$$

The length of the rod is $5a$.

Thus the distance measured parallel to the rod of the centre of gravity of the chains from O is $\frac{11}{16}$ of the length of the rod.

The centre of gravity of each chain is clearly at its middle point, and the middle points of all the chains lie on a straight line through O.

Hence the centre of gravity required is the point in which the vertical from a point $\frac{11}{16}$ of the length of the rod from O cuts the line through O which bisects all the chains.

39. Properties of the Centre of Gravity.

PROPOSITION 37. *A body can be divided into two portions, the centre of gravity of each of which is known, to find the centre of gravity of the whole.*

Let W_1, W_2 be the weights of the two portions, G_1, G_2 their centres of gravity.

Join G_1G_2, Fig. 88, and divide it in G so that

$$W_1 . GG_1 = W_2 . GG_2.$$

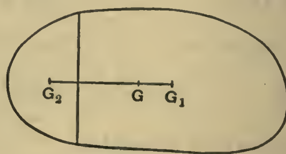


Fig. 88.

Then G is the centre of gravity of weights W_1, W_2 at G_1 and G_2; thus it is the point required.

Moreover by adding $W_1 . GG_2$ to both sides we get

$$W_1(GG_1 + GG_2) = (W_1 + W_2) GG_2.$$

Hence
$$GG_2 = \frac{W_1 . G_1G_2}{W_1 + W_2},$$

and
$$GG_1 = \frac{W_2 . G_1G_2}{W_1 + W_2}.$$

PROPOSITION 38. *Having given the centre of gravity of a body and of a portion of the body, to find that of the other portion.*

Let W be the weight of the whole body and G, Fig. 88, its centre of gravity, let W_1 be the weight of one portion, G_1 its centre of gravity. Join G_1G and in G_1G produced take a point G_2 such that

$$W . G_2G = W_1 . G_2G_1,$$

then G_2 is the point required.

For subtract from each side of the equation the value $W_1 . G_1G_2$,

then $$(W - W_1)\, G_2G = W_1\,(G_2G_1 - G_2G)$$
$$= W_1 . G_1G_2.$$

Hence G is the centre of gravity of W_1 at G_1 and $(W - W_1)$ at G_2.

But $W - W_1$ is the weight of the second portion of the body and hence G_2 is its centre of gravity.

Examples. (1) *A lamina has the form of a square with an isosceles triangle attached to one side. The side of the square is* a, *the height of the triangle is* h, *find the position of the centre of gravity of the figure, and determine the value of* h *if it lie in the base of the triangle.*

Let $ABCD$ (Fig. 89) be the square, ABE the triangle, G_1 the centre of gravity of the square, G_2 of the triangle. The line G_1G_2 passes through E and bisects AB at right angles.

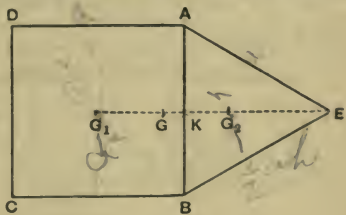


Fig. 89.

Let it cut AB in K, and let G be the centre of gravity of the figure.

The mass of the square is proportional to a^2, that of the triangle to $\frac{1}{2}ah$.

Thus the whole mass is proportional to $a^2 + \frac{1}{2}ah$.

The distance $$KG_1 = \tfrac{1}{2}a,$$
and $$KG_2 = \tfrac{1}{3}h.$$
Hence $$G_1G_2 = \tfrac{1}{2}a + \tfrac{1}{3}h.$$

The moment about any point of the whole weight at G is equal to the sum of the moments about the same point of a^2 at G_1 and $\frac{1}{2}ah$ at G_2.

Take moments about K.

Then $$(a^2 + \tfrac{1}{2}ah)\, KG = a^2\, KG_1 - \tfrac{1}{2}ahKG_2$$
$$= \frac{a^3}{2} - \frac{1}{2}\,\frac{ah^2}{3}.$$

Thus
$$KG = \frac{3a^2 - h^2}{3\,(2a + h)},$$
which determines the position of the point required.

If G is in the line AB then $KG = 0$.

Hence $$h^2 = 3a^2,$$
and $$h = a\sqrt{3}.$$

(2) *A portion of a square lamina is removed by a line which passes through the middle points of two adjacent sides; find the centre of gravity of the remainder.*

Let $ABCD$ be the square, G its centre of gravity, E and F the middle points of AB and AD. G_1 the centre of gravity of the part AFE.

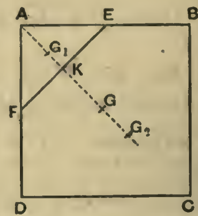


Fig. 90.

Then GG_1 passes through A and bisects FE at right angles. Let K be the point of intersection. Let $GA=a$ so that $2a$ is the diagonal of the square.

Then $$KG=KE=KA=KF=\frac{a}{2}.$$

$$KG_1=\tfrac{1}{3}KA=\frac{a}{6}.$$

Hence $$GG_1=\tfrac{2}{3}a.$$

Also the area of the square is $2a^2$, and the area of the triangle is $\frac{a^2}{4}$.

Thus the area left when the triangle is removed is

$$2a^2-\frac{a^2}{4},\text{ or }\tfrac{7}{4}a^2.$$

Also its centre of gravity is on G_1G produced, let it be G_2.

Then, taking moments about G,

$$GG_1\cdot\frac{a^2}{4}=GG_2\frac{7a^2}{4}.$$

Hence $$GG_2=\tfrac{1}{7}GG_1=\tfrac{2}{21}a.$$

(3) *From a square, whose side is* a, *an isosceles triangle of altitude* h *is removed, the side of the square being the base of the triangle; find the centre of gravity of the remainder.*

Let AEB (Fig. 91) be the triangle, $ABCD$ the square, G the centre of gravity of the square, G_1 of the triangle.

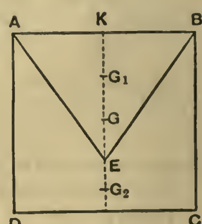


Fig. 91.

EGG_1K is a straight line bisecting AB in K, and the centre of gravity of the figure formed by removing the triangle lies in G_1G produced; let it be G_2.

Area of the square $=a^2$.

Area of the triangle $=\tfrac{1}{2}ah$.

Area of figure formed by removing the triangle

$$=a^2-\tfrac{1}{2}ah=\tfrac{1}{2}a(2a-h).$$
$$KG=\tfrac{1}{2}a,\quad KG_1=\tfrac{1}{3}h.$$

Hence $GG_1 = \frac{1}{2}a - \frac{1}{3}h = \frac{3a-2h}{6}$.

Take moments round G.

Then $$\tfrac{1}{2}ah \,.\, GG_1 = \tfrac{1}{2}a\,(2a-h)\,GG_2.$$

Thus $$GG_2 = \frac{h}{2a-h}\,GG_1 = \frac{h\,(3a-2h)}{6\,(2a-h)}.$$

Thus the position of G_2 is found.

The results of these last three examples may be verified by experiment by cutting out in stiff card or sheet metal lamina of the shape indicated, and determining the positions of their centres of gravity by the method indicated in Experiment 4.

40. Equilibrium of a body resting on a horizontal surface. When a heavy body rests on a flat horizontal surface it is in equilibrium under its weight, which acts vertically downwards through its centre of gravity, and the upward pressures at the points of contact with the surface.

These upward pressures have a vertical resultant, and this resultant must balance the weight; it must therefore act vertically upwards in a line which passes through the centre of gravity of the body. If the form and position of the body is such that this is impossible the body cannot be in equilibrium.

Thus suppose the body is a vertical lamina, a sheet of card or metal having the shape shewn in Fig. 92, and that it is in contact with the table at two points A and B. The pressures of the table at A and B are both upward vertical forces. The line of action of their resultant must lie between A and B. Hence for equilibrium the position of the centre of gravity must be such that a vertical drawn through it must fall between A and B. If the centre of gravity be in a position such as G, Fig. 92, the lamina can remain in equilibrium, if it have a position such as G in Fig. 93 equilibrium is impossible.

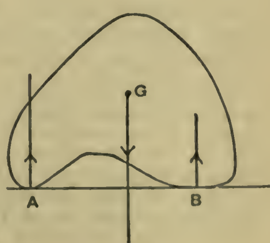


Fig. 92.

In the same way if the body rest on three points, like the legs of a three-legged table, the resultant of the upward pressures balances the weight. If a triangle be formed by joining the points in which the three legs rest on the floor, the resultant upward pressure must act within this triangle, therefore the vertical through the centre of gravity must fall within the triangle.

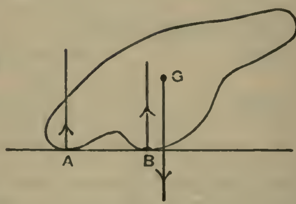


Fig. 93.

Or suppose again that the body rests in contact with the table at more points than three. Imagine a string drawn tightly round the body so as to include all these points of contact, thus forming a closed polygon whose sides are either straight or convex outwards. The pressures at the various points of support all act vertically upwards at points within the area thus defined; the resultant pressure therefore acts vertically upwards at some point within this area, and this resultant pressure, since it balances the weight, must pass through the centre of gravity.

Thus if equilibrium is possible the vertical through the centre of gravity must fall within the area thus defined. This area is sometimes spoken of as *the base of the body*, and the proposition is expressed in the statement that, in order that a body, resting on a plane under gravity, may be in equilibrium, it is necessary that the vertical through the centre of gravity should fall within the convex polygon formed by joining the extreme points of contact of the body and the plane.

This polygon must have no reentrant angles. Thus if $ABCDE$ (Fig. 94) be points of contact having a reentrant angle at D, the boundary of the base is $ABCEA$. A string stretched round the points of support would pass from C to E. The resultant pressure must lie within this polygon, though it might quite well lie without the polygon $ABCDE$.

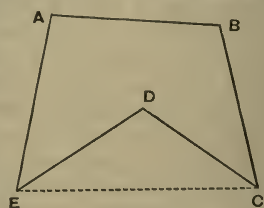


Fig. 94.

Examples. (1) *A right-angled triangle ABC having angles at B and C of* $30°$ *and* $60°$ *respectively, rests in a vertical plane on a horizontal table, the side AC being vertical and A being the right angle.*

The point C is joined to a point D in AB and the triangle DAC is removed. Find the largest triangle which can thus be removed without disturbing the equilibrium of the rest.

Bisect BC (Fig. 95) in L and join DL. The centre of gravity of the triangle CBD lies in DL. If the angle BDL is acute, a vertical from the centre of gravity must fall within the "base" BD, if it be obtuse the vertical must fall outside the "base," the limiting position of D then will be found by drawing LD vertical, and in this case LD is parallel to CA. Thus since L is the middle point of BC, D is the middle point of BA. Hence the area of the triangle DBC = the area of the triangle DAC $= \frac{1}{2}$ area of the triangle ABC, and half the original triangle may be removed without disturbing the equilibrium.

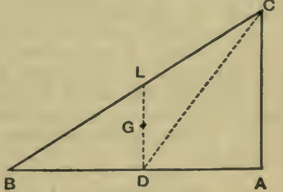


Fig. 95.

(2) *A brick $8 \times 3 \times 4$ inches in size rests with its smallest face on an inclined plane, the 3-inch side being horizontal: the brick is prevented by the friction from slipping down. Find the greatest angle to which the plane can be raised without causing the brick to fall over.*

Let $ABCD$ (Fig. 96) be a section of the brick by a vertical plane, AB being the inclined plane. Let G be the centre of gravity of the brick.

Then so long as the vertical through G falls between A and B, equilibrium is possible. In the limiting position GA is vertical. Thus the angle GAD is equal to the angle of the plane.

Now AG passes through C.

$$\text{Thus } \tan GAD = \tan CAD = \frac{CD}{AD}$$

$$= \frac{4}{8} = \frac{1}{2}.$$

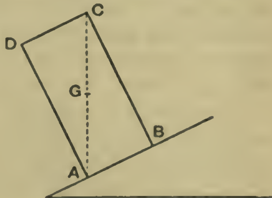


Fig. 96.

Thus the plane can be raised until it makes with the horizon an angle whose tangent is $\frac{1}{2}$.

(3) *A circular table rests on three legs attached to three points in the circumference at equal distances apart. A weight is placed on the table, determine in what position the weight is most likely to upset the table, and find the least value of the weight which when placed in that position will upset the table.*

If the table is upset by placing a weight on it, it will at first turn round an axis passing through the lower ends of two of the legs. A given weight therefore will be most effective in turning the table over when its

moment round such an axis is greatest. This will be the case when the weight is as close to the edge of the table as possible, and at a point D midway between two of the legs A and B.

Again let ABC (Fig. 97) represent the top of the table and G its centre of gravity, A, B, C being the points at which the legs are attached, and D the middle point of the arc AB. Join GD cutting the line AB in K. Then if W be the weight of the table, W' that of the weight which is placed on it, the table will not upset so long as the moment of W' about AB is not greater than that of W. Hence in the limiting condition we must have $W'.DK = W.GK$.

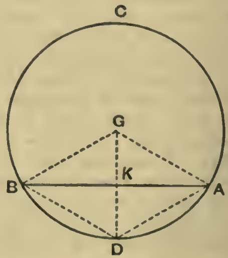


Fig. 97.

But since the angle AGB is 120° and DA is equal to DB, the triangles DGA, DGB are equilateral and the figure $DAGB$ is a rhombus, thus DG is bisected in K.

Hence $DK = KG$.

Thus the table will not upset so long as the weight supported at D is less than that of the table.

41. Stability of Equilibrium. Consider any body supported at one point, such as a lamina, which can turn round a horizontal axis through a point O, Fig. 98. We have seen that the condition for equilibrium is that the centre of gravity should be in the vertical through O. Three cases however may occur; the centre of gravity may be below O, or above O, or it may coincide with O.

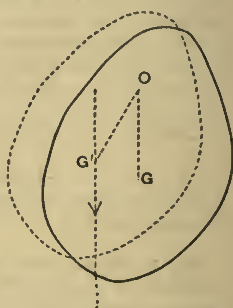


Fig. 98.

Consider the first case and suppose the body to be slightly displaced so that the centre of gravity G is brought to G'. Then the weight of the body W acting through G' has a moment about O which tends to bring the body back to its original position; in this case the equilibrium is said to be *stable*.

If however G be above O as in Fig. 99, and the body be displaced so that G may come to G' the moment of the weight

about O tends still further to increase the displacement, the equilibrium is *unstable*.

And thirdly if O and G coincide the body will balance in any position however it may be turned about O, the equilibrium is said to be *neutral*. The above illustration affords an example of what is meant by the terms **stable**, **unstable** and **neutral**, which are applicable generally to bodies in a position of equilibrium.

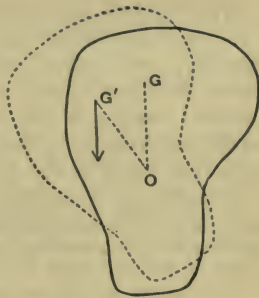


Fig. 99.

Definition. *Consider a body which has been slightly displaced from a position of equilibrium. If the body tends to return to that position, its equilibrium is stable.*

Thus a weight suspended by a string from a fixed point is in stable equilibrium, so is an egg resting with its shortest diameter vertical, or a sphere which has been loaded at one point and rests on the table with the loaded part downwards.

Definition. *A body at rest which, after receiving a small displacement, tends to move further away from its equilibrium position is in unstable equilibrium.*

Thus it is possible to make an egg rest on its point, or to balance a stick on its lower end, but the very slightest disturbance upsets the equilibrium; again a loaded sphere may rest with the load uppermost, but if ever so little displaced it will turn until the load comes to the bottom. These are all cases of unstable equilibrium. A wheel which has a load attached at one point can rest with this load either below or above the axle. In the second position the equilibrium is unstable, if the wheel be disturbed the load will move until it settles itself in the lowest position.

Definition. *A body is in neutral equilibrium when after receiving a small displacement it will rest in its new position.*

A truly balanced wheel or a uniform sphere or cylinder resting on a flat surface are all in neutral equilibrium.

Now in these various cases, in which the weight of the body is the only impressed force in addition to the reaction of the supports; we notice, that for equilibrium the centre of gravity is either as high as possible or as low as possible. The potential energy of the body depends on the position of its centre of gravity; in the first case the potential energy has a maximum value, in the second case it has a minimum value.

In either case, if the body be very slightly displaced, the height of the centre of gravity, and therefore the value of the potential energy, is at first altered by a quantity which is itself very small compared with the change in the position of the body.

Thus let the body be turned through a very small angle θ, so that OG (Fig. 100) becomes OG', and $\angle GOG' = \theta$. Draw $G'K$ perpendicular to OG, then the centre of gravity is raised a distance GK.

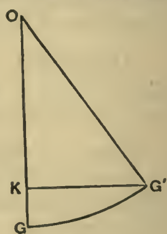


Fig. 100.

Now if $\angle GOG'$ is very small, then $OG'G$ is very nearly a right angle. Hence the triangles KGG' and $G'GO$ are similar.

Thus $$\frac{KG}{GG'} = \frac{GG'}{GO}.$$

Hence $$KG = \frac{GG'^2}{GO}.$$

And if GG' is small GG'^2/GO is very small indeed. Compared with its horizontal displacement, the change in height of the centre of gravity is very small.

This proposition is found always to be true, in an equilibrium position the potential energy has always a maximum or a minimum value, the change in potential energy consequent on a small displacement is very small when compared with the displacement, it depends on the square of the displacement.

Again, in all the cases of unstable equilibrium, the centre of gravity is as high as possible, the potential energy has a maximum value, the change produced by a small displacement is very small but, such as it is, it tends to reduce the potential energy of the body, the energy tends to take the kinetic form, the potential energy tends to decrease.

When the equilibrium is stable the centre of gravity is as low as possible, the potential energy has, in the equilibrium position, a minimum value, the change due to the displacement, though very small, is an increase; to produce it, work must be

done against the impressed forces, the body must gain energy; this cannot take place without a supply of energy from without, hence a position of rest in which the potential energy is a minimum is a stable position, for to displace the body work must be done on it from without.

In the unstable position some of the potential energy can be transformed into kinetic, and this is a change which will go on of itself if once started by a small displacement, the position therefore is unstable, the body can do work on external bodies if properly connected with them and when once started will do that work.

If the body in the unstable position be quite free from all impressed force except its weight, and the (frictionless) reaction at the point of support, it acquires kinetic energy in falling as fast as it loses potential energy, it therefore passes through the position of stable equilibrium with an amount of kinetic energy equal to the potential energy it has lost, and this, if there were no friction and no air resistance, would carry it up on the other side to the former unstable position; in practice, however, some of the energy is dissipated as heat and in other ways; the body does not rise to the unstable position but comes instantaneously to rest before reaching it. It then falls back through the stable position and continues to oscillate about this until the kinetic energy it has acquired in the fall is all dissipated, when it comes finally to rest in this position with a minimum of potential energy.

Thus a position of minimum potential energy is one of stable equilibrium because work must be done to displace the body from this position, and to do this work needs a supply of energy from without.

A position of maximum potential energy is one of unstable equilibrium because it is possible for some of the potential energy to be transformed into kinetic energy without an external supply, and this is a change which in nature can take place of itself.

We do not know why there is this tendency for the transformation of energy or by what process it goes on, all that we observe is that motion which can take place without gain of

energy to the system, and which merely involves transformation of energy will occur, while motion which involves a gain of energy will not occur unless energy be communicated from without.

EXAMPLES.

1. A cylinder, 50 ft. long, balances on a log put under it at 30 ft. from one end, and it also balances on the log put under its centre, when a weight of 50 lb. is placed at one end and 120 lb. at the other; find the weight of the cylinder.

2. Find by a diagram the centre of gravity of 3 cylindrical rods, of unequal lengths but small uniform thickness, so placed as to form a triangular figure. Where is the centre of gravity, in this case, geometrically situated?

3. A circular table weighing 20 lb. is supported by vertical legs attached to 4 points of the rim forming a square; find from what parts of the rim a hundredweight can be hung without overturning the table.

4. Find the centre of gravity of a lamina formed by a square having a part cut off by means of two cuts reaching from the centre to the two adjacent corners.

5. Twelve equal heavy particles are placed round the circumference of a circle at equal distances from each other. Two of the particles which have three particles between them are now removed; find the centre of gravity of the remaining ten particles.

6. If from a uniform lamina in the form of an equilateral triangle of side a the triangular portion formed by joining the middle points of two of its sides is cut away; find the distance of the centre of gravity of the remaining piece from the centre of gravity of the whole triangle.

7. From a body whose centre of gravity is known a portion whose centre of gravity is known is cut away. Find the centre of gravity of the remaining piece.

If the body is a uniform lamina in the form of a rectangle, and the triangle formed by joining its centre of gravity G to the ends of one of the sides is cut away; find the distance of the centre of gravity of the remaining part from the point G, where a is the length of the adjacent side.

8. A parallelogram $ABCD$ weighs 3 lb. and is divided by its diagonal BD into two parts, one of which, viz. the triangle BCD, is twice as heavy as the other. If a weight of 1 lb. is placed at the corner A of the parallelogram, find the centre of gravity of the system.

9. A regular hexagon is inscribed in a circle, and weights of 1 lb. each are placed at 5 of the angular points of the hexagon, and 3 lb. at the centre of the circle. Find the centre of gravity of the system.

10. If three equal triangles are cut off a triangle by lines respectively parallel to the three sides, shew that the centre of mass of the remaining figure coincides with that of the original triangle.

11. From a square piece of paper $ABCD$ a portion is cut out in the form of an isosceles triangle whose base is AB and altitude equal to one third of AB. Find the centre of gravity of the remaining portion.

12. A table with a heavy square top $ABCD$ rests upon four equal and heavy legs, placed at A, B, E and F, where E and F are the middle points of BC and CD. Shew that the table will be upset by a weight upon it at C, just greater than the weight of the whole table; and find the greatest weight which may be placed at D without upsetting the table.

13. A circle of radius r touches internally at a fixed point, a fixed circle of radius R; find the centre of gravity of the area between them, and its ultimate position when r increases and becomes ultimately equal to R.

14. The middle points of two adjacent sides of a uniform rectangular lamina are joined and the lamina is cut in two along the joining line. Find the centre of gravity of the larger portion.

15. From a body, weight W, a piece of weight w is cut and moved a distance x; shew that the centre of gravity of the whole moves a distance $xw \div W$ in the same direction.

$ABCD$ is a trapezium, the angles at B and C being right angles. Shew that the distance of its centre of gravity from BC is

$$\frac{AB^2 + AB \cdot CD + CD^2}{3(AB + CD)}.$$

16. A triangle is cut off from a uniform square plate by a section along a line joining the middle points of two adjacent sides. Will it be possible to balance the remainder in a vertical position with one of the sides that has been cut in contact with a horizontal plane?

17. ABC is a triangle; forces represented by $3AB$ and $4AC$ act along the sides AB and AC. Prove that their resultant cuts BC at a point G, such that $BG = \frac{4}{7}BC$.

18. A rod whose length is 10 feet, and which is thicker at one end than at the other, balances about its centre when 10 lb. is hung from one end and 20 from the other; while if 40 lb. instead of 20 is hung from the second end the fulcrum is at 4 feet from that end. Find the weight of the rod and the position of its centre of gravity.

19. A uniform rod of weight W is supported from a point by two strings. One of these makes an angle of 60°, the other an angle of 30° with the rod. Find the tensions in the strings.

20. A thin square board whose weight is 1 lb. has one quarter of one edge resting on the end of a horizontal table, and is kept from falling over by a string attached to an upper corner of the board and to a point on the table in the same vertical plane as the board. If the length of the string be double that of the edge of the board, find its tension.

21. A beam 12 feet long rests on two supports, distant 2 feet from each end. The beam weighs 1 cwt. Find the greatest weight which can be supported from one end without overbalancing the beam. Find also the pressure on each support when this weight is suspended.

22. A circular hole 1 foot in radius is cut out of a circular disc 3 feet in radius. If the centre of the hole be 18 inches from that of the disc, find the centre of gravity of the remainder.

23. Where must a circular hole of 1 ft. radius be punched out of a circular disc of 3 ft. radius, so that the centre of gravity of the remainder may be 2 inches from the centre of the disc?

24. Two isosceles triangles are on the same base but on opposite sides of it, and the altitude of one is 6 inches and of the other 2 inches. Find the distance from the common base of the centre of gravity of the whole figure.

25. A cylinder of wood 12 inches long is 4 inches in diameter for 8 inches of its length, and 3 inches in diameter for the remaining 4 inches. Determine the position of its centre of gravity.

26. ABC is a triangle, whose sides AB, BC, CA are 6, 10 and 8 inches long: at A, B and C respectively, are weights of 7, 8 and 9 lb. Shew that the centre of gravity of the weights coincides with that of the perimeter of the triangle.

27. The middle points of two adjacent sides of a uniform triangular lamina are joined and the lamina is cut in two along the joining line. Find the centre of gravity of the larger portion.

28. How, practically, may the centre of gravity of a heavy beam be found of which one end is heavier than the other? If it be made up of two uniform cylinders whose lengths are as 3 : 5, and weights as 3 : 1, where is the centre of gravity?

29. A uniform bar 4 yards long weighing 12 lb. has three rings each weighing 6 lb. upon it at distances 1 foot, 5 feet and 7 feet from one end. At what point will it balance?

30. One corner of a square is cut off by a straight line passing through the middle points of two adjacent sides. Find the position of the centre of gravity of the remainder.

31. A uniform triangular plate hangs from one angle with the base horizontal; shew that the triangle is isosceles.

CHAPTER VI.

MACHINES.

42. Simple Machines. There are various contrivances by which the amount or the direction of a force impressed on a body can be modified. By impressing a small force at one point of a body we may be able to give rise to a large force acting, it may be, in a different direction at some other point; or *vice versa*, some small force may be produced through the action of a larger force impressed elsewhere.

A contrivance for either of these purposes is called a machine.

DEFINITION. *An apparatus for making a force impressed on a body at a given point and in a given direction available at some other point or in some other direction is called a Machine.*

We should notice at the outset that through the action of a machine the force exerted may be greater than that applied, yet it follows from the principle of energy, that no more work is done by the machine than is done on it. Energy supplied to the machine at one point is transmitted by it to some other point; the amount of such energy, except for frictional loss, remains unchanged.

The words "except for frictional loss" are of course important, for in nature there is very considerable loss in any machine. More work must be done on the machine than it can do.

Now any machine such as a pump, a steam-engine or a crane, consists of a number of simple parts, these we shall find it desirable to classify and deal with separately.

Each of these parts is spoken of as a **Simple Machine.**

We shall suppose further that the machine is in equilibrium, and that a single force impressed at one point just balances another force impressed in general at some other point.

We look upon the first force as exerted by some other body on the machine, it is often called the **Power**, though the name is not a good one, for power means rate of doing work. The second force we consider as a force which the machine exerts on some other body, this is ordinarily described as the **Weight.** In addition to these we have the reactions at the points of support of the machine.

The names Power and Weight come from the fact that the simplest machines were no doubt originally devices to enable men to raise weights. By means of certain contrivances a man is able, though he can exert but little force, to raise a heavy weight; the force he can exert measures the power, the weight he raises is the weight.

The simple machines consist in all cases of bodies which are constrained so as to be capable of motion in some definite manner, two forces applied to such a body balance and the problem is to find the relation between them.

Now we suppose the machines to be frictionless, and the fundamental principle which will apply to all is that if the machine be supposed to receive any very slight possible displacement, the work done by the one force just balances that done against the other.

If then we measure the distance which the point of application of each force is displaced in the direction of the force, and multiply that displacement by the corresponding force, the two products will be numerically equal. Work is done by one force and against the other, the amount of work in each case being the same. If one force be large and the other small, they may balance, but in this case the displacement of the point of application of the first is small, that of the second is large.

Work is measured by the product of two factors, Force and Displacement. In many cases it is convenient to change it from the form of a small force multiplied by a large displacement into the form of a large force multiplied by a small displacement. A machine enables this to be done.

If P, Q are the two forces which balance on a machine, p, q the corresponding displacements of the points at which the forces are impressed measured parallel to the lines of action of P and Q, then

$$P \cdot p = Q \cdot q.$$

Hence
$$\frac{p}{q} = \frac{Q}{P},$$

or the displacements are inversely as the corresponding forces.

This result is sometimes expressed by the statement that "what is gained in Power is lost in Speed." If the "Power" P be small and the "Weight" Q large, then p is large and q is small, so that, in order to raise a large "Weight" by the aid of a small "Power," the point at which the "Power" is applied must move through a large distance compared with that traversed by the weight.

In most machines the relation between the "Power" and the "Weight" can be found most simply by making use of this principle; the problems which occur will however be solved in this way and also by the direct application of the conditions of equilibrium of a system of forces. We shall thus obtain verifications of the principle for the simple machines.

DEFINITION. *When two forces, a "Power" and a "Weight," impressed on a machine maintain it in equilibrium, the ratio of the weight to the power is called the* **Mechanical Advantage of the Machine.**

The reason for the name is clear; the object of most machines is to balance a large "Weight" with a small "Power." When this can be done mechanical advantage is gained by the use of the machine.

The Simple Machines may be classified as:

(i) The Lever, including the Wheel and Axle.

(ii) The Pulley.

(iii) The Inclined Plane, including the Wedge.

(iv) The Screw.

43. The Lever. The Lever is a rod or bar which may be either straight or curved and which can move only about a fixed point.

This point is called the fulcrum, and is denoted by C in the figures.

A force P applied at one point A balances another force W applied at a second point B; we wish to find the relation between the two. The lines of action of the two forces and of the reaction at the fulcrum must lie in one plane and meet in a point or be parallel; we will take this plane as the plane of the paper.

The conditions of equilibrium are obtained from the same principle in all cases. The resultant of the forces P and Q impressed at A and B, Fig. 101, must pass through C. Hence the moment of P round C must be equal to that of Q about C. Thus if CL, CM be perpendicular from C on the lines of action of the forces $P . CL = Q . CM$.

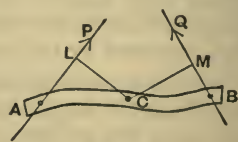


Fig. 101.

We will now consider a little more in detail the various cases which arise, and in the first place we will deal with a straight lever and suppose the forces P and Q to be parallel and at right angles to the length of the lever.

Proposition 39. *To find the mechanical advantage of the straight lever.*

Levers of this kind are usually divided into three classes.

Class I. The points A and B at which the Power and the Weight are applied, Fig. 102 (a), are at opposite ends of the lever and the fulcrum C is between them. Levers of this class are a crowbar as ordinarily employed to raise a weight, the beam of a balance, a pair of scissors, or the handle of an ordinary pump.

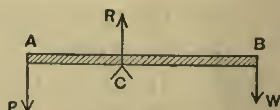


Fig. 102 (a).

For the conditions of equilibrium we have if R be the pressure on the fulcrum and a, b the lengths of the arms CA,

CB respectively,

$$R = P + W,$$

$$P \,.\, CA = W \,.\, CB.$$

Hence $$\frac{W}{P} = \frac{CA}{CB} = \frac{a}{b}.$$

Thus the mechanical advantage may be greater or less than unity according as the Power acts at the end of the longer or the shorter arm.

Class II. The Power and the Weight act on the same side of the fulcrum C but in opposite directions, the Power being applied at a greater distance from the fulcrum than the Weight.

Among levers of this class are an oar and a pair of nut-crackers. In the case of the oar, the portion of the blade in the water is the fulcrum, the power is applied by the oarsman, the pressure of the rowlock corresponds to the weight, the fulcrum is of course not absolutely fixed.

For this class then we have Fig. 102 (b)

$$R = W - P,$$

$$P \,.\, CA = W \,.\, CB.$$

Hence $$\frac{W}{P} = \frac{CA}{CB} = \frac{a}{b}.$$

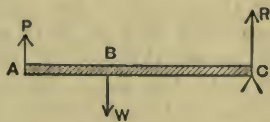


Fig. 102 (b).

And since a is greater than b the mechanical advantage is always greater than unity.

Class III. The Power and the Weight act in opposite directions as in Class II., but the Power is nearer the fulcrum than the Weight.

As examples we have some forms of the treadle of a lathe or sewing machine, or a pair of spring shears, the blades of which are held open by a spring and are closed by the pressure of the hand applied at a point between the spring and the blade.

Another important example is the bone of the forearm, the fulcrum is the elbow joint, the power is applied by a muscle

attached to the arm not far from the joint, the weight being held in the hand.

In this case we have, Fig. 102 (c),

$$R = P - W,$$

$$P \,.\, CA = W \,.\, CB.$$

Therefore

$$\frac{W}{P} = \frac{CA}{CB} = \frac{a}{b},$$

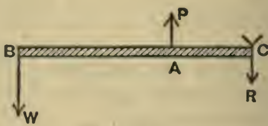


Fig. 102 (c).

and since a is less than b the mechanical advantage is less than unity, a small weight is raised by a large power, but the point of application of the power moves over a small distance while the weight is considerably displaced.

44. Bent Levers. If the lever be not straight, or the forces P and W be not parallel, we still find their ratio by taking moments round the fulcrum.

Again let the directions of P and W, Fig. 103, meet at O, then R is the resultant of P and W at O and it passes through C, thus if γ be the angle between OA and OB, R acts along OC while its value is given by

$$R^2 = P^2 + W^2 + 2PW \cos \gamma.$$

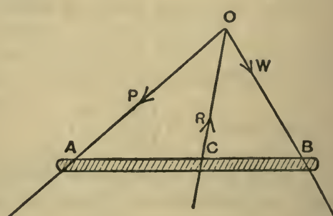


Fig. 103.

We may obtain an equation to find the direction of R thus, supposing ACB to be straight.

Let $OAB = \alpha$, $OBA = \beta$, and let $OCB = \theta$.

Then R acts along OC, P and W along OA and OB respectively.

Resolve the forces perpendicular to AB.

$$R \sin \theta = P \sin \alpha + W \sin \beta \quad \text{.........................(1)}.$$

Resolve parallel to AB.

$$R \cos \theta = P \cos \alpha - W \cos \beta \quad \text{.........................(2)}.$$

Squaring and adding we have

$$R^2 = P^2 + W^2 - 2PW\cos(\alpha+\beta).$$

Dividing (1) by (2)

$$\tan\theta = \frac{P\sin\alpha + W\sin\beta}{P\cos\alpha - W\cos\beta}.$$

In the above equations we have not taken into account the weight of the lever; this can if necessary be done. Assuming the forces all to be vertical, we have to add the weight of the lever to the pressure on the fulcrum and include the moment of the weight applied at the centre of gravity in the equation of moments.

45. Application of the Principle of Work. We can readily obtain the relation between the power and the weight for the lever by an application of the principle of work.

This has already been done in the general case in Section 22, for it was proved there that when a body can turn about an axis the work done by any force is found by multiplying the moment of the force by the circular measure of the small angle turned through.

Hence if the work done is zero the sum of the moments of the forces is zero and this applied to two forces gives us the principle of the Lever. The ratio of the two forces is equal to the inverse ratio of the arms at which they act. We will however apply the principle of work to the case of a straight lever on which two parallel forces are impressed at right angles to the arms.

Let ACB, Fig. 104, be the lever, ACB being a straight line,

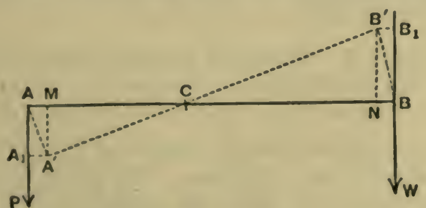


Fig. 104.

let P, W be the forces impressed at the points A and B respectively in directions perpendicular to the lever.

Let $$CA = a, \quad CB = b.$$

Let the lever be turned about C through a small angle θ into the position $A'CB'$. Draw $A'A_1$, $B'B_1$, perpendicular to the directions of P and W respectively, and $A'M$, $B'N$ perpendicular on ACB.

Then $$A_1A = A'M,$$
and $$B_1B = B'N.$$

Work done by $P = P \cdot AA_1 = P \cdot A'M$.

Work done against W
$$= W \cdot BB_1 = W \cdot B'N.$$

Hence, since these two amounts of work are equal, we have
$$P \cdot A'M = W \cdot B'N.$$

Thus $$\frac{P}{W} = \frac{B'N}{A'M} = \frac{A'C}{B'C},$$
for the triangles $A'CM$, $B'CN$ are similar.

Also $$A'C = AC = a,$$
$$B'C = BC = b.$$

Hence $$\frac{W}{P} = \frac{a}{b},$$
which is the result required.

We may also obtain this result by direct experiment. The bar employed has already been described, Section 18, and is shewn in Figure 105.

EXPERIMENT 6. *To find by experiment the relation between the Power and the Weight in a lever and to verify the law that the work done by the Power is equal to that done on the Weight.*

You are given a straight graduated bar ACB, Fig. 105, moveable about C as a fulcrum. This point is very approximately coincident with the centre of gravity of the bar, which will therefore balance about C. The weight of the bar is thus

directly supported by pressure at the fulcrum and need not be further considered. Rings A and B slide on the bar and

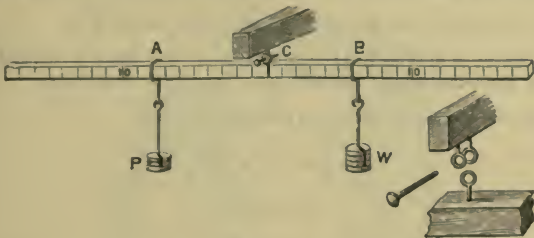


Fig. 105.

from these rings respectively weights which we will call P and W are supported. Place A with its weight P in any convenient position on the bar and adjust either the weight W or the position of the ring B until the bar rests in equilibrium in a horizontal position. Measure the distances AC and BC. Then it will be found that $P \times AC = W \times BC$. Again, measure the heights of A and B above the floor or some other convenient horizontal plane. Then lower the end A and fix the bar in an inclined position $A'CB'$. Measure the heights of A' and B' and hence determine the distances a and b say, through which the Power has been lowered and the Weight raised. It will be found that $P \times a = W \times b$, or the work done by the power is equal to that done on the weight.

Various forms of balances exemplify in their action the principle of the lever. These will be dealt with in a separate section. (See § 59.)

Examples. (1) *Weights of* 10 *and* 15 *lb. are suspended from the ends of a lever* 12 *feet in length; find the point at which they balance.*

Let AB (Fig. 106) be the lever, C the fulcrum, the upward pressure at C is 25 lb.

Let the 10 lb. weight be at A.

Take moments about A.

$$25 \,.\, AC = 15 \,.\, 12,$$

$$AC = \frac{15 \,.\, 12}{25} = \frac{3 \,.\, 12}{5} = 7\tfrac{1}{5} \text{ ft.}$$

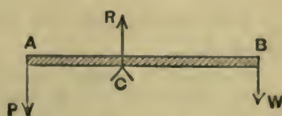


Fig. 106.

(2) *A straight rod is loaded so that its centre of gravity is $\frac{1}{3}$ of its length from one end. When weights of 5 and 10 lb. are supported from the ends, the rod balances about its middle point; find the weight of the rod.*

Let the length of the rod be l feet and let it weigh W lb.

The centre of gravity is $\frac{1}{6}l$ from the centre and the 5 lb. weight clearly hangs on the same side of the centre as the centre of gravity.

Hence, taking moments about the middle point,

$$5 \cdot \frac{l}{2} + W \cdot \frac{l}{6} = 10 \cdot \frac{l}{2}.$$

Thus $$W = 15 \text{ lb.}$$

(3) *Two weights P and Q are suspended from points A and B in a straight rod of weight W. The rod can move about a fulcrum C. If A, C, B and the centre of gravity G be in a straight line and the rod be in equilibrium when inclined at an angle θ to the horizon, shew that it will be in equilibrium in any other position.*

Let the direction of P, W and Q meet a horizontal line through C in M, L and N (Fig. 107) respectively.

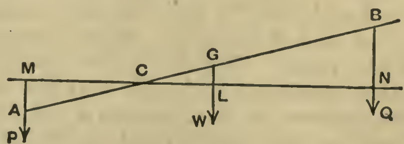


Fig. 107.

Take moments about C.

Then $$P \cdot CM = W \cdot CL + Q \cdot CN.$$

Now $$CL = CG \cos\theta,\ \ CM = CA \cos\theta,\ \ CN = CB \cos\theta.$$

Hence $$P \cdot CA \cos\theta = W \cdot CG \cos\theta + Q \cdot CB \cos\theta.$$

Thus dividing out by $\cos\theta$ we have

$$P \cdot CA = W \cdot CG + Q \cdot CB.$$

Since this relation does not involve the angle θ it will be true for all values of θ.

It should be noticed however that $\cos\theta$ must not be zero, if it were we should not be justified in dividing out by it. Thus in the initial position the rod must not be vertical; clearly if it were vertical it would be in equilibrium whatever the weights might be.

46. The Wheel and Axle. The apparatus is shewn in Fig. 108. It consists of a wheel or drum of considerable

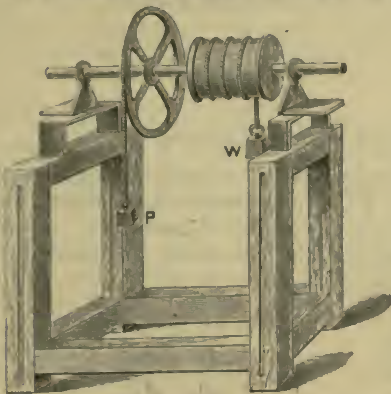


Fig. 108.

diameter round which a rope can be coiled and which can turn about an axis through its centre. A string coiled on this wheel carries the power P. A drum of smaller diameter—the axle—is mounted on the same axis. Round this a rope is coiled in the opposite direction to the first and carries the weight W. Thus when P is lowered the rope round the wheel is uncoiled, that round the axle is coiled up and W is raised.

Proposition 40. *To find the mechanical advantage of the Wheel and Axle.*

Fig. 109 represents a plan of the wheel and axle, perpendicular to the axis round which it can rotate. It is clear that the machine acts like a lever. C the centre is the fulcrum, CA, CB radii of the wheel and axle respectively are the arms.

Let $CA = a$, $CB = b$.

Then for the conditions of equilibrium we have *either*

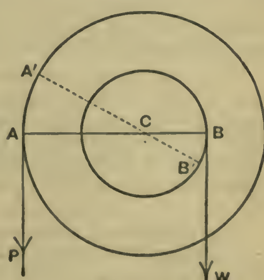


Fig. 109.

(i) By taking moments about C,

$$P\,.\,CA = W\,.\,CB.$$

Thus
$$\frac{W}{P} = \frac{CA}{CB} = \frac{a}{b},$$

or

(ii) By the principle of work.

Let the apparatus be turned through a small angle θ so that the line $A'CB'$ may become horizontal. Then clearly the power P drops a vertical distance AA', an amount AA' of rope is uncoiled, while W is raised a distance BB'. An amount of rope BB' is coiled up.

Hence
$$W\,.\,BB' = P\,.\,AA'.$$

Therefore
$$\frac{W}{P} = \frac{AA'}{BB'},$$

and since ACB and $A'CB'$ are straight lines

$$\frac{AA'}{BB'} = \frac{AC}{BC} = \frac{a}{b}.$$

Hence as before

$$\frac{W}{P} = \frac{a}{b}.$$

In the above we have treated the rope as though it were a mathematical line of no thickness. In experiments it may quite well happen that the thickness of the rope is comparable with the radius of the axle, if this is so we may suppose the power and the weight to act respectively at the centre of the rope, and we have then to add to the radii, both of the wheel and of the axle, half the thickness of the rope.

Since the mechanical advantage of the wheel and axle is given by the ratio a/b, we could, by making a large and b small, raise by means of a small power a very large weight, were it not for the fact that if b be too small the axle will not be sufficiently strong to carry the weight. We cannot reduce b beyond a certain limit without endangering the machine. This difficulty is avoided in the differential wheel and axle. See Section 55.

The mechanical advantage of the wheel and axle can be determined by experiment by finding the weight which a given power can support. Friction will however probably prevent any very close agreement between experiment and theory.

47. The Pulley. The Pulley is a small circular disc or wheel with a groove cut in its outer edge round which a string can pass. The wheel can turn on an axis through its centre, the ends of this axis are carried by the block within which the pulley turns.

When the block is fixed as in Fig. 110, the pulley is said to be fixed, in other cases it is moveable. The weight is attached to one end of a string which passes over the groove of the pulley; the power in the case of a fixed pulley can be applied at the other end of the string. If, as we shall suppose, the supports of the pulley are smooth the tension at all points of the string must be the same throughout, and the power will then be equal to the weight.

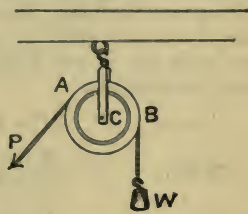


Fig. 110.

For consider the two points A, B, where the string leaves the pulley, and let C be the centre, P the power applied at A, W the weight suspended from B. Then, taking moments about C, we have

$$P \,.\, CA = W \,.\, CB.$$

But $$CA = CB.$$

Hence $$P = W.$$

The fixed pulley is useful only in changing the direction of a force.

48. The single moveable pulley. In this instrument the weight W is suspended from the pulley block; the string passes round the pulley, one end of the string is secured as at C to a fixed support, the power P is applied upwards as at A.

In this case as in others the strings may either be parallel or inclined to each other.

PROPOSITION 41. *To find the relation between the Power and the Weight in a single moveable pulley.*

(i) *When the strings are parallel.*

The forces acting are the tensions of the two parallel strings AD and BC, Fig. 111, and the weight; the weight acts vertically, hence the two strings are vertical. Moreover the tensions are equal and each is equal to P. Thus we have $2P$ upwards balancing W, which acts downwards.

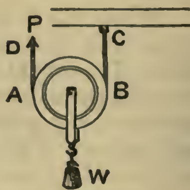


Fig. 111.

Hence $\qquad 2P = W.$

Thus $\qquad \dfrac{W}{P} = 2,$

or a given power can raise twice its own weight.

(ii) *When the strings are not parallel.*

Since the tensions in the two strings are equal and are balanced by the weight, their resultant is equal and opposite to the weight, but the resultant of two equal forces bisects the angle between the forces. Thus the two strings are equally inclined to the vertical, let θ, Fig. 112, be the angle between either string and the vertical. The tension in each string is equal to P, hence resolving vertically

$$2P\cos\theta = W.$$

Thus $\qquad \dfrac{W}{P} = 2\cos\theta.$

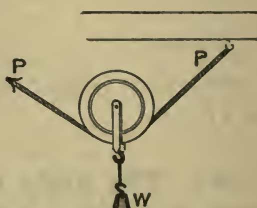


Fig. 112.

If the weight of the pulley be w and it be sufficient to be considered then the downward vertical force is $W + w$, and the last equation becomes

$$\frac{W+w}{P} = 2\cos\theta.$$

If the "Weight" be not vertical but be applied in some other direction by means of a string or rope attached to the pulley block, then its direction still bisects the angle between the strings as in Fig. 113, and a similar equation holds.

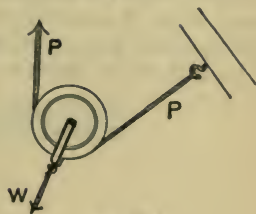


Fig. 113.

Proposition 42. *To apply the Principle of Work to a single moveable pulley with parallel strings.*

Suppose the pulley raised a distance x, so that its centre may move from O to O'.

The simplest way of doing this, Fig. 114, is to suppose both ends of the string to be raised an equal distance x from C and D to C' and D' respectively, the work done, since the tension in each string is P, is $2Px$.

Now suppose that C' is lowered to C again, the pulley being kept fixed, D' will rise an equal distance x to D'', but no work will be done by this, for the work done at one end of the string just balances that done in the other. Thus the end D at which the power P is applied is raised $2x$ and the total work done is as above equal to $2P \cdot x$.

Hence $$P \cdot 2x = W \cdot x.$$

or $$\frac{W}{P} = 2.$$

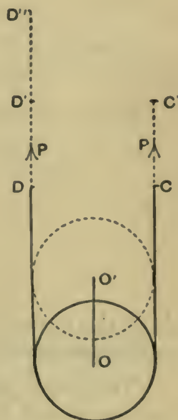


Fig. 114.

If the strings are not parallel we have two equal forces at a point balanced by a third and the problem is the same as that solved in Section 33.

49. Systems of Pulleys. Various combinations of Pulleys are in common use. Some of these will be described.

Proposition 43. *To find the mechanical advantage of the first system of pulleys.*

The first system of pulleys consists of a number of pulleys, each of which is suspended by a separate string. One end of each string is attached to a fixed support, the other end of the string after passing round a pulley is fastened as shewn in Fig. 115, to the block of the next pulley. The weight is hung

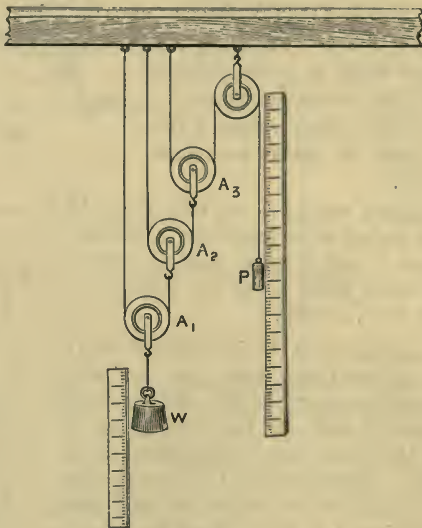


Fig. 115.

from the lowest pulley, the string from the highest moveable pulley passes over a fixed pulley and to it the power is applied.

Thus each pulley after the lowest is acted on downwards by its weight and by the tension of the string which connects it to the pulley next below, and upwards by the two tensions in the parts of the string by which it is supported. Thus the tension in the string round any pulley is half the sum of the weight of that pulley and the tension of the string below it.

Thus let $w_1, w_2, w_3 \ldots$ be the weights of the moveable pulleys $A_1, A_2, \ldots$ etc., $t_1, t_2, t_3 \ldots$, the tensions of the strings round A_1, A_2, etc., and suppose there are n moveable pulleys, then

$$t_1 = \frac{1}{2}(W + w_1)$$

$$t_2 = \frac{1}{2}(t_1 + w_2) = \frac{1}{2^2}(W + w_1) + \frac{1}{2}w_2$$

$$t_3 = \frac{1}{2}(t_2 + w_3) = \frac{1}{2^3}(W + w_1) + \frac{1}{2^2}w_2 + \frac{1}{2}w_3,$$

and so on.

Moreover $$t_n = P.$$

Hence

$$P = t_n = \frac{1}{2^n}(W + w_1) + \frac{1}{2^{n-1}}w_2 + \dots \frac{1}{2^{n-2}}w_3 + \dots \frac{1}{2}w_n,$$

or multiplying up by 2^n,

$$2^n P = W + w_1 + 2w_2 + 2^2 w_3 + \dots 2^{n-1} w_n.$$

If the weights of the pulleys be neglected the expressions become simpler though the principle is just the same.

Thus, if there be four moveable pulleys, the tension in the first string is $\frac{W}{2}$, in the next $\frac{W}{2^2}$, and in the fourth $\frac{W}{2^4}$.

Hence $$\frac{W}{2^4} = P.$$

Therefore $$\frac{W}{P} = 2^4 = 16.$$

Thus with this system a given "Power" P could support a "Weight" of $2^n . P$ where n is the number of moveable pulleys.

Proposition 44. *To apply the Principle of Work to the first system of pulleys.*

Let the weight and the first pulley rise a distance x. The end of the string round this pulley rises $2x$; thus the second pulley rises $2x$, the next pulley rises twice this or 2^2x. Thus if there be n pulleys as before the "Power" P moves a distance $2^n x$.

Hence

$$P \,.\, 2^n x = (W + w_1)\,x + w_2 \,.\, 2x + w_3 \,.\, 2^2 x + w_n \,.\, 2^{n-1}\,x.$$

Therefore

$$2^n P = (W + w_1) + 2w_2 + 2^2 w_3 + \dots\ 2^{n-1} w_n,$$

which is the same result as previously obtained.

PROPOSITION 45. *To find the mechanical advantage of the second system of pulleys.*

In the second system there are two sheaves of pulleys in separate blocks. The string is attached to one of the blocks, Fig. 116,—in the figure it is the upper—and passes round the pulleys in turn first under one in the lower block, then over one

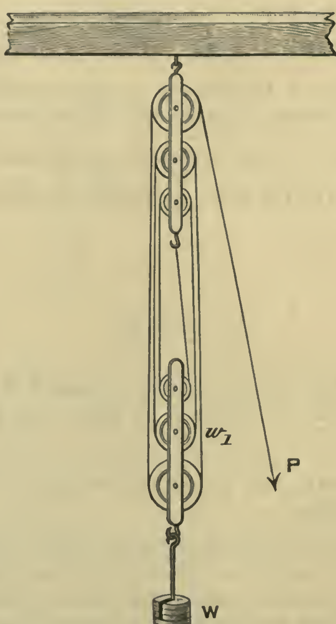


Fig. 116.

in the upper and so on. The pulleys are sometimes arranged with a common axis, sometimes the various pulleys in a block are placed one below the other as in the figure.

In either case the tension of the string is equal to the power; let there be n strings at the lower block. The upward force will be found by multiplying the tension by n, the downward force is the weight supported W, together with the weight of the lower block w_1.

Hence $$nP = W + w_1.$$

We can apply the principle of work thus. If the lower block be raised a height x, a length x of each string will be left slack. Hence the end of the string can move a distance nx.

Thus, $$P \cdot nx = (W + w_1)\, x,$$

or $$nP = W + w_1.$$

PROPOSITION 46. *To find the mechanical advantage of the third system of pulleys.*

In this system, Fig. 117, one end of each string is attached

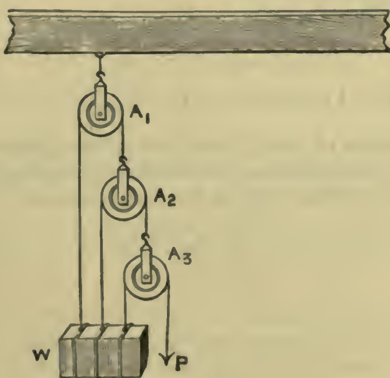


Fig. 117.

to a bar which carries the weight. The uppermost pulley is fixed; a string passes over it and supports the next pulley, another string passes over this and supports the third, and

so on, the last string passes over the last moveable pulley and to it the Power is applied.

Now if t_1, t_2 .. t_n be the tension in the strings beginning from that over the topmost pulley, w_1, w_2 ... the weights of the pulleys :

Then
$$t_n = P,$$
$$t_{n-1} = 2t_n + w_n = 2P + w_n,$$
$$t_{n-2} = 2t_{n-1} + w_{n-1} = 2^2 P + 2w_n + w_{n-1},$$
$$t_1 = \dots 2^{n-1} P + 2^{n-2} w_n + \dots + w_2.$$

Also
$$\begin{aligned} W &= t_1 + t_2 + \dots t_n \\ &= P(1 + 2 + 2^2 + \dots + 2^{n-1}) \\ &\quad + w_n(1 + 2 + 2^2 \dots 2^{n-2}) \\ &\quad + w_{n-1}(1 + 2 \dots + 2^{n-3}) \\ &\quad + \dots w_2. \end{aligned}$$

Now we know that
$$1 + 2 + 2^2 + \dots + 2^{r-1} = 2^r - 1.$$

Thus
$$W = P(2^n - 1) + w_n(2^{n-1} - 1) + w_{n-1}(2^{n-2} - 1) + \dots + w_2.$$

If the weights of the pulleys be neglected the expression is simplified, for the tension in each string, beginning from the power, is clearly twice that in the string before.

Thus
$$t_1 = P, \ t_2 = 2P \dots$$
$$t_n = 2^{n-1} P.$$
$$\begin{aligned} W &= t_1 + t_2 + \dots t_n \\ &= P\{1 + 2 + 2^2 + \dots 2^{n-1}\} \\ &= P\{2^n - 1\}. \end{aligned}$$

We may apply the Principle of Work thus. Let the weight and the bar carrying it rise a distance x.

If all the pulleys retained their positions fixed there would be a length x slack in each string.

In consequence of this alone each pulley after the first could be lowered a distance x; the first pulley is fixed, hence the second pulley drops a distance x; in consequence of this drop the third pulley will be lowered through twice this distance or $2x$; to this must be added the direct drop x due to the rise of the weight, which would have taken place even if the second pulley had been fixed.

Thus the actual drop of the third pulley is $2x+x$, or $3x$, and we may write this $(2^2-1)\,x$.

In consequence of this the fourth pulley drops twice as far or $2\,(2^2-1)\,x$, and to this we must add the fall x for the direct rise of the weight.

Thus the drop of the fourth pulley is

$$2\,(2^2-1)\,x+x,$$

or

$$x\,(2^3-2+1),$$

or

$$x\,(2^3-1).$$

The law is now clear, the nth pulley drops $x\,(2^{n-1}-1)$, and the Power

$$x\,(2^n-1).$$

Thus the equation of work gives

$$Wx=Px\,(2^n-1)+w_nx\,(2^{n-1}-1)+\dots$$
$$+\dots+w_3x\,(2^2-1)+w_2x.$$

Therefore

$$W=P\,(2^n-1)+w_n\,(2^{n-1}-1)+\dots$$
$$+w_3\,(2^2-1)+w_2.$$

50. Experiments with Pulleys. It is not possible to verify by direct experiment these results. In any system of pulleys the friction is considerable, a smaller power than that given by the equations is sufficient to support a given weight, a larger power than is given is necessary just to raise the weight.

We can however verify by direct measurement the result that when a pulley rises a distance x, and one end of the string round it is held fixed, then the other end rises a distance $2x$.

Thus consider the first system of pulleys. Support a Weight W from the lowest pulley, and let the Power P be another weight supported in a suitable scale-pan. Adjust two vertical scales as shewn in Fig. 115 above, by the side of the power and the weight respectively. Displace the system by

raising the weight W a measured distance a on its scale, and observe the distance through which P descends, it will be found to be equal to 2^nP where n is the number of moveable pulleys. Similar observations can be made for the other systems of pulleys.

In the third system of pulleys the bar to which the weight is attached will not remain horizontal unless the point of attachment of the weight is the point of action of the resultant of the tensions. Thus if K be the point of attachment, A the end of the bar to which the string carrying the weight is attached, and $2a$ the distance between the strings, that is the diameter of each pulley, then taking moments about K

$$W \,.\, AK = t_1 \,.\, 0 + t_2 \,.\, 2a + t_3 \,.\, 4a + \ldots$$

By substituting the values of t_1, t_2, etc., found in Section 49, the position of K can be found.

51. The Inclined Plane. Consider a plane inclined to the horizon at an angle α. A weight could be raised to the top of such a plane—if we neglect friction or take means to make it small—with the application of a smaller force by causing it to slide up the plane, than by lifting it directly. Hence the inclined plane is one of the mechanical powers.

The solution of the problem depends in part on the direction in which the force is impressed.

PROPOSITION 47. *To find the mechanical advantage of an inclined plane when the Power acts parallel to the plane.*

Let BAC, Fig. 118, be an inclined plane, AC being horizontal and BC vertical. Let the angle BAC be equal to α, and consider a body of weight W at rest on the plane, which is supposed to be smooth. Let AB, the length of the plane be equal to l, and let the height BC be h.

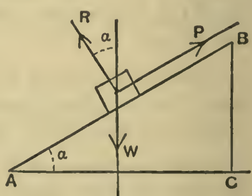


Fig. 118.

The forces acting are the weight W vertical, the power P parallel to the plane, and the resistance R at right angles to the plane. The relation between these quantities can be found in various ways.

(i) *By the resolution of forces.*

The direction along the plane, in which P acts, and a line at right angles to this, along which R acts, will clearly be "convenient" directions in which to resolve the forces.

The angle between the normal to the plane—the direction of R—and the direction of W is clearly α. Hence the component of W, perpendicular to the plane, is $W\cos\alpha$, and along the plane it is $W\sin\alpha$.

Hence resolving along the plane

$$P = W\sin\alpha,$$

and perpendicular to the plane

$$R = W\cos\alpha.$$

Thus P and R are found in terms of W and α.

Moreover, $$\sin\alpha = \frac{BC}{AB} = \frac{h}{l}.$$

Therefore $$\frac{P}{W} = \sin\alpha = \frac{h}{l},$$

or $$P\,.\,l = W\,.\,h.$$

(ii) *By an application of the principle of work.*

The work done by P in moving the body a small distance s along the plane is $P\,.\,s$, if at the same time the body rise a height z, the work done against z is $W\,.\,z$.

No work is done by the resistance R, since the motion is everywhere at right angles to the direction of R.

Hence $$P\,.\,s = W\,.\,z,$$

or $$\frac{W}{P} = \frac{s}{z} = \frac{h}{l},$$

from the figure.

Hence $W\,.\,h = P\,.\,l$, as before.

We can obtain the result directly by considering the work done in moving from A to B.

(iii) *By the triangle of forces.*

Since the forces P, W and R, maintain the body in equilibrium, they must be proportional to the sides of any triangle drawn parallel to them. Let G, Fig. 119, be the particle. Let GK, vertical, meet AC in K, and GL at right angles to the plane meet AC in L. Draw KM parallel to the plane to meet GL in M. Then P, R and W are respectively parallel to KM, MG and GK.

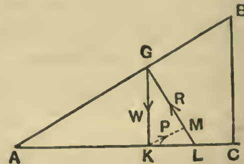


Fig. 119.

Hence $\dfrac{P}{KM} = \dfrac{R}{MG} = \dfrac{W}{GK}$.

Again the triangles CBA, MKG are similar.

Hence
$$\frac{KM}{GK} = \frac{BC}{AB} = \sin\alpha,$$
$$\frac{MG}{GK} = \frac{AC}{AB} = \cos\alpha.$$

Thus
$$\frac{P}{W} = \frac{KM}{GK} = \frac{BC}{AB} = \frac{h}{l} = \sin\alpha,$$
$$\frac{R}{W} = \frac{MG}{GK} = \frac{AC}{AB} = \cos\alpha.$$

PROPOSITION 48. *To find the mechanical advantage of an inclined plane when the power acts horizontally.*

Let AB, Fig. 120, be the plane, BC being vertical and AC horizontal.

Let P be the power acting horizontally, and let α be the angle of the plane. Let $BC = h$ and $AB = l$.

Let W be the weight and R the resistance of the plane.

We can find the mechanical advantage in various ways as follows.

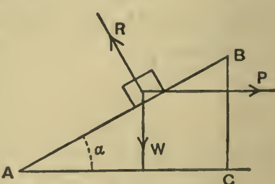


Fig. 120.

(i) *By the resolution of forces.*

Resolve horizontally and vertically. The components of R are $R\cos\alpha$ vertical, and $R\sin\alpha$ horizontal. P is horizontal and W vertical.

Hence, resolving vertically,

$$W = R\cos\alpha.$$

Resolving horizontally

$$P = R\sin\alpha.$$

Thus $$\frac{P}{W} = \frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha.$$

(ii) *By the principle of work.*

In moving the body from A to B, since the displacement in the direction of P is AC, while that opposite to the direction of W is CB, the work done by P is $P.AC$, and that done against W is $W.BC$.

Thus $$P.AC = W.BC.$$

Hence $$\frac{P}{W} = \frac{BC}{AC} = \tan\alpha.$$

(iii) *By the triangle of forces.*

Let G, Fig. 121, be the body, and let GK vertical meet AC in K, and GL normal to the plane meet it in L.

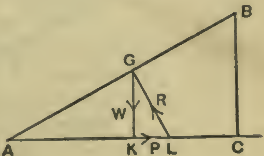


Fig. 121.

Then P, R and W are parallel respectively to KL, LG and GK.

Thus $$\frac{P}{KL} = \frac{R}{LG} = \frac{W}{GK}.$$

But the triangles LKG and BCA are similar.

Hence $$\frac{P}{W} = \frac{KL}{GK} = \frac{BC}{AC} = \tan\alpha,$$

$$\frac{R}{W} = \frac{LG}{GK} = \frac{AB}{AC} = \sec\alpha.$$

If the force be inclined as shewn in Fig. 122 at an angle ϵ to the plane, it is usually simplest to resolve parallel and perpendicular to the plane.

Since three forces which maintain a body in equilibrium are in one plane and R and W lie in a vertical plane through the particle, the direction of P must also be on this vertical plane.

Resolving parallel to the plane we have

$$P \cos \epsilon = W \sin \alpha.$$

Resolving perpendicular to the plane we have

$$R + P \sin \epsilon = W \cos \alpha.$$

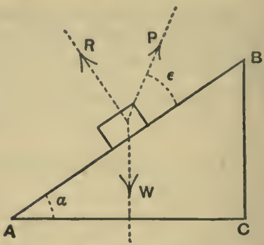


Fig. 122.

Some of these results can be verified by experiment.

Experiment 7. *To prove that on an inclined plane, when the Power acts parallel to the plane, the ratio of the Power to the Weight is equal to that of the height of the plane to its length, and to verify the Principle of Work.*

The plane is a wooden board shewn at AB, Fig. 123, to which a sheet of glass is attached. The board is hinged at A to a second board, which can be clamped to the table. At B, which is at some convenient distance (say 10 inches from A),

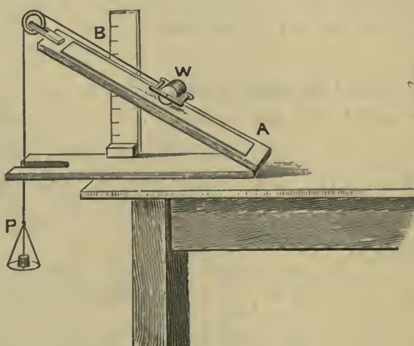


Fig. 123.

there is a thumb-screw. By means of this there can be clamped to the plane a vertical rod with a slot parallel to its length. The rod is graduated in such a way that the height of B above the lowest point A can be read off directly.

Thus the height h can be measured, and the length l is a known constant. A pulley is fixed at the top of the plane. The "Weight" W consists of a heavy brass roller mounted in a frame so as to turn with very little friction, a string attached to the frame passes over the pulley and supports a scale-pan, into which various weights can be placed. The "Power" P is the weight of this scale-pan and weights.

The frame and pulley are arranged so that the string between them when tight is parallel to the plane. Thus the Power P acts on the Weight W parallel to the plane.

Set the plane so that h may have some convenient value, say 5 inches. Observe the value of P required to support W for this value of h. To do this accurately find the value, P_1, which will just drag W up the plane and the value, P_2, which will only just let it roll down. The proper value of P, that is, the value which it would have if there were no friction, is the mean of P_1 and P_2. When the observations are made it will be found that $W . h = P . l$, or that $P : W = h : l$. Again, if α is the angle which the plane makes with the horizon $h/l = \sin \alpha$. Hence $P = W \sin \alpha$. In our case since the length l is 10 inches we have to divide the height by 10 to get $\sin \alpha$, thus if $h = 5$ inches, $\sin \alpha = \cdot 5 = \frac{1}{2}$, and $\alpha = 30°$. Again, suppose the weight is allowed to move down the plane, it will have fallen a vertical height of h inches. Thus the work done by the weight will be $W . h$ units of work; and clearly the power P, since it is attached to W by the string, must have been raised l inches and the work done on it will be Pl units; but by what we have seen $P . l = W . h$, which proves the principle of work for the inclined plane in this case. Take a series of values of P for various values of h and thus shew that in all cases $P . l = W . h$.

52. The Wedge. This is a sort of double inclined plane or prism, Fig. 124, made of iron or steel or some such

hard material and used for splitting wood or for other like purposes.

Thus if BAC, Fig. 125, be a wedge of angle α driven into a piece of wood by a weight W applied downwards, and we

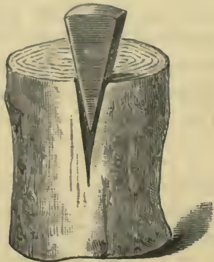


Fig. 124.

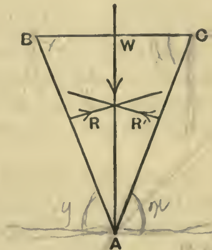


Fig. 125.

y+z
B+C+

suppose R, R', the pressures which the two faces of the wedge exert on the obstacle, to act normally to its faces AB and AC, and further that these two faces are equally inclined to the vertical, at angles therefore of $\frac{\alpha}{2}$, we can find the relation between R, R' and W, thus, supposing the wedge to be smooth.

The vertical components of R and R' balance W, the horizontal components of R and R' are in equilibrium.

Hence resolving vertically

$$W = (R + R') \sin\frac{\alpha}{2},$$

$$R \cos\frac{\alpha}{2} = R' \cos\frac{\alpha}{2}.$$

x = C
y = B
B+C+A = 2∂

Hence $$R = R',$$

and $$W = 2R \sin\frac{\alpha}{2}.$$

In reality the friction involved in the use of the wedge is enormous.

We ought to consider two other forces F, F' acting parallel to the faces of the wedge.

If we suppose the machine to be symmetrical with regard to the vertical line bisecting the angle α, we have

$$W = 2R\sin\frac{\alpha}{2} + 2F\cos\frac{\alpha}{2};$$

and unless we know the relation between R and F we cannot carry the solution any further.

53. The Screw. Consider a sheet of paper cut into the form of a right-angled triangle BAC, Fig. 126 (a), and wrap it round a cylinder so that the base AC of the triangle

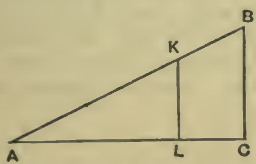


Fig. 126 (a).

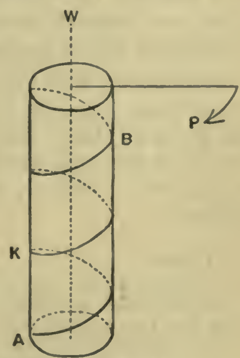


Fig. 126 (b).

may be at right angles to the axis of the cylinder. The hypotenuse AB will form a spiral curve round the cylinder, Fig. 126 (b). Now imagine a projecting thread to be fixed to the outside of this cylinder so as to coincide with the spiral curve thus drawn; the cylinder then becomes a "screw." Suppose now that the paper is wound inside a hollow cylinder of the same radius as the solid cylinder just described and that a hollow groove is cut in the surface of this hollow cylinder, the groove being of such a form that the projecting thread just fits it. The hollow cylinder constitutes a "nut" in which the

screw can turn, Fig. 127. Let the nut be held fixed and the end of the screw inserted in it. Turn the screw round its axis by means of a lever; as it is turned its end moves outwards through the nut parallel to the axis. If the axis be vertical, a weight W can be raised by the application of a power P to the end of the lever. In any case if the nut be held fixed, force can be exerted by the end of the screw.

Fig. 127.

The angle α which the thread of the screw makes with a plane at right angles to the axis is called the angle of the screw.

The distance measured parallel to the axis between two consecutive turns of the thread is called the pitch and depends on the angle and on the radius of the cylinder on which the screw is cut.

For let K, Fig. 126 (a), be a point on the paper triangle which when it is rolled on to the cylinder comes directly over the point A. Draw KL perpendicular to the base AC, parallel that is to the axis, when the paper is rolled on the cylinder, L will coincide with A, and KL is the distance between two threads.

Hence $$KL = h.$$

Again AL is clearly the circumference of the cylinder so that if b be its radius we have

$$AL = \text{circumference of a circle of radius } b = 2\pi b.$$

But $$\frac{KL}{AL} = \tan KAL = \tan \alpha.$$

Thus $$KL = AL \tan \alpha,$$

or $$h = 2\pi b \tan \alpha.$$

We notice further that when the screw makes one complete revolution, a point such as K is brought into the position previously occupied by A, the end of the screw advances a distance h parallel to the axis.

It is impossible to obtain a screw in which there is no friction. We must therefore suppose that at each point of the thread there is acting a normal force at right angles to the thread, and a tangential force parallel to it. Suppose these forces to be uniformly distributed, and let R and F be their values for each unit of length of the screw. Let l be the whole length of the screw. R may be resolved into a force $R\cos\alpha$ parallel to the axis, and $R\sin\alpha$ at right angles to it; and F has for its components $F\sin\alpha$ parallel to the axis, $F\cos\alpha$ at right angles to it. The forces at right angles to the axis have moments round the axis, the others have not.

Let us suppose also that we are trying to raise the weight, then the frictional force helps the action of the weight and opposes that of P.

Suppose further that P acts at the end of an arm a in a direction at right angles to the axis.

Then resolving vertically

$$W = Rl\cos\alpha + Fl\sin\alpha.$$

Taking moments about the axis

$$Pa = Rl\sin\alpha \,.\, b + Fl\cos\alpha \,.\, b.$$

If W and P are known these equations will give us R and F; we cannot use them to find the mechanical advantage unless we have some relation between F and R.

If we suppose F is zero, which in practice is never the case, then

$$W = Rl\cos\alpha,$$

$$Pa = Rl\sin\alpha \,.\, b.$$

Hence $$\frac{W}{P} = \frac{a}{b}\cot\alpha.$$

PROPOSITION 49. *To find the mechanical advantage of the Screw.*

We can obtain the result most easily by the Principle of Work. For while W is being raised a distance h, the point of application of P moves once round a circle of radius a, hence its displacement is $2\pi a$; moreover the direction of P is tangential to this circle. Thus the work done is $P \,.\, 2\pi a$.

Hence $$P \,.\, 2\pi a = Wh = W \,.\, 2\pi b\tan\alpha.$$

Thus $$\frac{W}{P} = \frac{2\pi a}{2\pi b\tan\alpha} = \frac{a}{b}\cot\alpha.$$

54. Combinations of Simple Machines. A complex machine is usually made up of a number of Simple Machines. In these the "Weight" of the first becomes the

"Power" of the next and so on. In such a case the mechanical advantage of the whole is found by multiplying together those of all the simple machines.

For let P_1, P_2, P_3, etc., be the powers, W_1, W_2, W_3, etc., the weights, m_1, m_2 ... m_n the mechanical advantages.

Then $$W_1 = P_2,\ \ W_2 = P_3 \dots W_{n-1} = P_n.$$

Hence
$$m_1 = \frac{W_1}{P_1} = \frac{P_2}{P_1},$$
$$m_2 = \frac{W_2}{P_2} = \frac{P_3}{P_2},$$
$$m_{n-1} = \frac{W_{n-1}}{P_{n-1}} = \frac{P_n}{P_{n-1}},$$
$$m_n = \frac{W_n}{P_n}.$$

Thus multiplying all together

$$m_1 \,.\, m_2 \,.\, m_3 \dots m_n$$
$$= \frac{W_n}{P_1} = \text{mechanical advantage of the whole.}$$

Some special forms of machines are described below.

55. The Differential Wheel and Axle. In this apparatus shewn in Fig. 128, the axle consists of two drums or cylinders of different radii, b and c. A rope, the ends of which are coiled in opposite directions round these two drums, passes under a single moveable pulley from which the weight W is supported. As the axle is turned the rope is coiled up on one drum and uncoiled from the other; the motion of the weight depends on the difference of these two effects. The power P is usually applied at the circumference of a wheel of radius a.

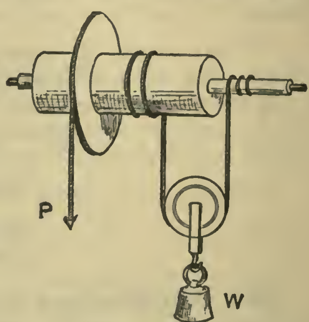


Fig. 128.

In Fig. 129, let $ACOB$ represent the machine as viewed from a point on the axis. Suppose it to be turned through a small angle θ so that $A'C'OB'$ may become horizontal.

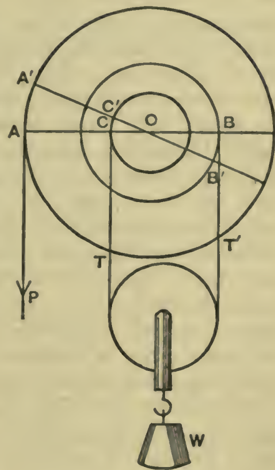


Fig. 129.

The power falls a distance $a\theta$, the end C of the string round the pulley falls a distance $c\theta$, while B rises a distance $b\theta$; hence the loop of string carrying the pulley is shortened by $(b-c)\,\theta$, and therefore the pulley and weight rise $\frac{1}{2}(b-c)\,\theta$.

Thus the Principle of Work gives

$$P \,.\, a\theta = W \,.\, \tfrac{1}{2}(b-c)\,\theta,$$

or
$$\frac{W}{P} = \frac{2a}{b-c}.$$

Hence by making b and c nearly equal, W can be made very large compared with P, without unduly reducing the strength of the machine.

For a given motion of the Power the distance traversed by the Weight can be made very small, hence the ratio of the "Weight" to the "Power" can be made very large.

We can solve the problem without using the Principle of Work thus:

Let T be the tension in the string carrying the pulley, then taking moments about O,

$$Pa + Tc = Tb.$$

Hence
$$Pa = T(b-c).$$

But from the equilibrium of the pulley $2T = W$.

Thus
$$P = \frac{T(b-c)}{a} = \frac{W(b-c)}{2a}.$$

56. The Differential Screw. This machine consists of two screws of slightly different pitch h and k. The axes of the two screws coincide, the second screw works inside the cylinder on which the first is cut. Thus if H, Fig. 130, be the outer screw, and K the inner screw, then on giving the outer screw one complete turn its point will move downwards through a distance h, and if the inner screw did not turn in the outer, it too would be displaced this same distance; but the inner screw does turn relatively to the outer, its point is in consequence raised relatively to the outer screw a distance k, thus the point of the inner screw actually descends a distance $(h-k)$. Hence if P be the "Power" impressed at one end of an arm a, we have

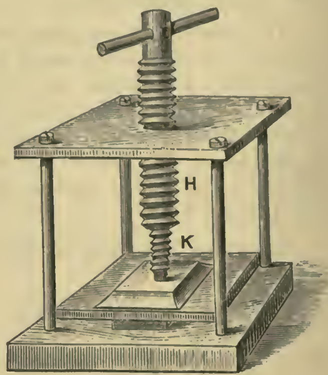


Fig. 130.

$$W(h-k) = P \,.\, 2\pi a.$$

Thus
$$\frac{W}{P} = \frac{2\pi a}{h-k}.$$

57. Cog Wheels. A train of cog wheels is virtually a combination of Wheels and Axles. Consider two such wheels, Fig. 131. Let A and B be their centres and let them be in

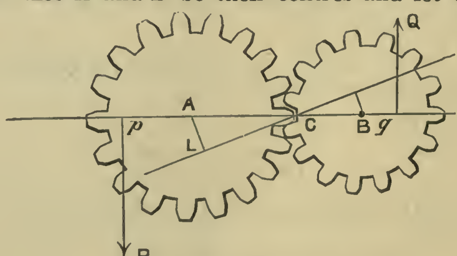


Fig. 131.

contact at C in the line AB. Let R be the force between the wheels at C, and let $AC=a$, $BC=b$. Draw AL, BM perpendicular on the direction of R.

Let the "Power" be a force P acting at an arm p, the "Weight" a force Q acting at an arm q.

Then for the equilibrium of A,

$$P \,.\, p = R \,.\, AL.$$

For the equilibrium of B,

$$Q \,.\, q = R \,.\, BM.$$

Also the triangles ACL, BCM are similar.

Thus $$\frac{P \,.\, p}{Q \,.\, q} = \frac{AL}{BM} = \frac{AC}{BC} = \frac{a}{b}.$$

Hence $$\frac{Q}{P} = \frac{p}{a} \,.\, \frac{b}{q}.$$

Now $\frac{p}{a}$ is the mechanical advantage of the wheel A considered as a bent lever; while $\frac{b}{q}$ is that of the other wheel. We have thus found the mechanical advantage of the whole, and see that it is the product of those of the two parts.

58. The Spanish Barton. This forms a useful combination of Pulleys shewn in Fig. 132. A and B are two moveable pulleys which are suspended by a string over a fixed pulley C: the "Weight" W is attached to A. A string passes from a fixed point D under A and over B, and the "Power" P is attached to this.

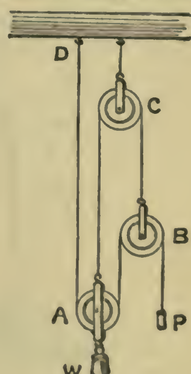


Fig. 132.

Let w_1, w_2 be the weights of the pulleys A and B.

Suppose the weight and the Pulley A raised a distance x. In consequence of this, if the pulley B were fixed, a length $2x$ of the string would be left slack. Thus P would fall a distance $2x$; but since A and B are connected by a string, when A

is raised a distance x, B falls the same distance. In consequence of this the "Power" falls a further distance $2x$: thus on the whole the "Power" falls $4x$. Hence W and A each rise x, P falls $4x$ and B falls x.

Therefore $$(W + w_1)\, x = w_2 x + P \,.\, 4x,$$
$$W = w_2 - w_1 + 4P.$$

In practice w_1 and w_2 would usually be equal. Hence the mechanical advantage (W/P) is 4.

We can also find the tension of the strings and solve the problem thus.

Let T be the tension in the string over the fixed pulley, that in the string round the moveable pulleys is P.

Hence for the equilibrium of A

$$W + w_1 = 2P + T,$$

and for that of B,

$$T = 2P + w_2.$$

Therefore $$W + w_1 = 4P + w_2.$$

59. The Balance. The balance, Fig. 133, in its simplest form consists of a lever which can turn about a fulcrum: it is used for comparing the masses of two bodies, or rather for

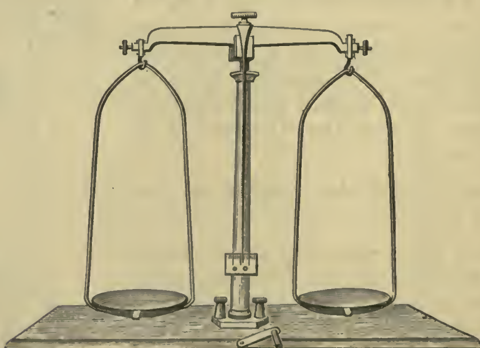


Fig. 133.

determining in terms of the standard mass the mass of any other body. This is done by comparing the weights of the bodies. From the arms of the lever two equal and similar scale-pans are suspended, the standard mass is placed in one of these, the body whose mass is required in the other, and, if the balance be in adjustment, the two masses are equal when the arms are horizontal.

To determine when this is the case with accuracy, a vertical pointer is attached to the beam near the fulcrum, the lower end of this pointer moves over a horizontal scale, being adjusted so as to rest at the centre of the scale when the beam is horizontal. The arms of the beam ought, as we shall see, to be equal and similar if the balance is to be accurate.

The requisites of a good balance are:

(i) Truth. (ii) Sensitiveness. (iii) Stability.

A balance is said to be *True* if the beam be horizontal whenever equal masses are placed in the scale-pans.

A balance is *Sensitive* when the beam deviates appreciably from its horizontal position for a very small difference $P - Q$, in the two masses P and Q.

A balance is *Stable* when the beam if disturbed from its equilibrium position readily comes back to it.

We shall consider how to secure these conditions separately. The points from which the scale-pans are suspended are A and B, Fig. 134, these in a good balance are steel knife-edges,

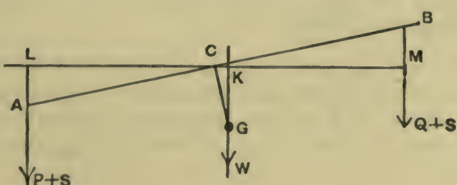


Fig. 134.

secured to the beam in such a way that their edges are at right angles to the length of the beam. The scale-pans are attached to small flat plates of steel, or better of agate, which

rest on these knife-edges when the beam is in use and hang with their centres of gravity below the respective knife-edges. The weights thus act vertically through A and B.

The fulcrum C is also a similar knife-edge resting on a steel or agate plate. In good balances there is an arrangement by which the plates are lifted off the knife-edges when the balance is not in use.

It is desirable[1] in a balance that the fulcrum C and the points of support of the scale-pans A and B should be in one straight line, and we shall assume this condition satisfied.

When the balance is loaded, the forces acting are the weight of the beam, let this be W, the weights of the scale-pans S, S' respectively and the weights P, Q of the masses in the scale-pans. The weight W acts at the centre of gravity of the beam; if the beam remains horizontal when the scale-pans are removed, this point must be vertically below the fulcrum, let it be G, then CG is at right angles to ACB.

Let $$CG = h, \ AC = a, \ BC = b.$$

PROPOSITION 50. *To find the condition that a balance may be true.*

The balance is true provided the beam be horizontal whatever equal masses are placed in the scale-pans; if the beam be horizontal the centre of gravity G is vertically below the fulcrum, thus the weight of the beam W has in this case no moment about the fulcrum.

Suppose now the scale-pans are empty, and the beam horizontal; the only forces which have a moment about the fulcrum are the weights S, S' of the scale-pans; these act vertically through A and B respectively. Hence taking moments about the fulcrum,

$$S \cdot a = S' \cdot b.$$

[1] It can be shewn that if this condition be not true the sensitiveness of the balance will vary with the load; the condition cannot be always accurately satisfied, for as the load increases the beam bends and the points A and B are brought down below the fulcrum.

Suppose now that two equal masses P, P are placed one in either scale-pan: if the balance be true the beam will still be horizontal, thus taking moments again

$$(P+S)\,a=(P+S')\,b,$$

or

$$P\,.\,a+S\,.\,a=P\,.\,b+S'\,.\,b.$$

But $$S\,.\,a=S'\,.\,b.$$

Hence subtracting $$P\,.\,a=P\,.\,b.$$

Thus $$a=b,$$

and since $S\,.\,a=S'\,.\,b$, we must have also

$$S=S'.$$

Therefore if a balance is to be true the arms must be equal in length and the scale-pans equal in weight.

PROPOSITION 51. *To find the condition that a balance may be sensitive.*

In finding this condition we assume that the balance is true. The condition required is that, when the weights P and Q differ slightly, the beam should be sensibly inclined to the horizon.

Let the centre of gravity of the beam be at G, a distance h below the fulcrum. Then if the beam is displaced, its weight W will have a moment about the fulcrum tending to restore it; in order that the displacement for a small difference $(P-Q)$ may be considerable the moment of the weight about C must be small; this moment will depend on the value of $W\,.\,h$, and can be made small by making W small or h small or both, thus for sensibility the beam must be light and its centre of gravity near the fulcrum.

But the displacement will also be great if the moment due to the difference of the weights is large.

This moment, since the balance is true, is $(P-Q)\,a$, and for a given difference $P-Q$ can be made large by making a large. Thus the sensitiveness is increased by lengthening the arms. The sensitiveness therefore may be made considerable either by (α) increasing the length of the arms, or (β) reducing the weight of the beam, or (γ) bringing the centre of gravity of the beam near the fulcrum.

In order to secure this last condition there is usually, in good balances, a short vertical wire attached to the beam above the fulcrum. This has a screw thread cut on it and a metal sphere can be screwed up or down on this wire; by raising the sphere the centre of gravity of the beam is raised and the sensitiveness increased.

We may put the results just obtained more briefly thus.

Let the beam be displaced through an angle θ and let a horizontal line through C meet in L, M and K, Fig. 134, the verticals through A, B and C, which are the lines of action of $P+S$, $Q+S$ and W respectively.

Then the angles ACL and CGK are both equal to θ.

And
$$CM = CL = CA\cos\theta = a\cos\theta,$$
$$CK = CG\sin\theta = h\sin\theta.$$

Thus taking moments about C,
$$(P+S)\,CL = (Q+S)\,CM + W\,.\,CK.$$

Hence
$$(P-Q)\,a\cos\theta = Wh\sin\theta,$$

thus
$$\frac{\tan\theta}{P-Q} = \frac{a}{W\,.\,h}.$$

Now $\tan\theta/(P-Q)$ is a measure of the sensitiveness. The balance is sensitive if this fraction be large when $P-Q$ is small. Thus for sensitiveness a/Wh is large.

Hence a must be large or Wh small, or both these conditions must hold.

PROPOSITION 52. *To find the condition that a balance may be stable.*

For this it is necessary that, when the pans are equally loaded, the beam after displacement should return rapidly to its position of equilibrium. Now if the pans be equally loaded and the balance be displaced the moments of the loads about the fulcrum balance, and the only moment tending to restore equilibrium is that of the weight of the beam. This depends on the product $W\,.\,h$; hence for stability this moment must tend to bring the beam back, thus G must be below the fulcrum, and for rapid action the product $W\,.\,h$ must be considerable.

It will not however be sufficient to make W large, for if this only is done the mass to be moved is increased as well as the impressed force and the acceleration is not changed. For great

stability then h must be considerable, the centre of gravity must be well below the fulcrum.

It will be noticed that this condition is antagonistic to one of those obtained for sensitiveness; fair sensitiveness and reasonable stability can be secured by making h not very small and giving the balance long arms.

The relative importance of the two conditions depends on the purposes for which the balance is required.

Stability and rapid action would be the main desideratum in a balance employed for weighing coals; a man conducting a chemical or physical research would attach greater importance to sensitiveness, though in this case also rapid action is important. A complete discussion of the question would take us beyond the limits of this book.

60. Use of a Balance. It is important to be able to test a balance for accuracy and to determine any errors which it may possess. Now in using a balance we have always to bring the beam back to the horizontal position. This may be done by placing weights in one or other of the pans until the pointer comes back to zero at the middle of the scale; instead of waiting however till the pointer is actually at the middle of the scale, we may notice the distance it moves on either side of the zero mark, and adjust the weights until these oscillations are equal, we then know that when the beam is at rest the pointer will be at zero[1].

We now proceed to describe some experiments with a balance.

EXPERIMENT 8. *To determine the ratio of the arms of a balance, and to weigh a body correctly in a balance with unequal arms.*

Let the balance come to rest with its pans unloaded, let S and S' be the weights of the scale-pans, a and b the lengths of the arms.

[1] A more exact method of "weighing by oscillation" is given in Glazebrook and Shaw's *Practical Physics*, Section 12, p. 107.

If the pointer does not come to zero, it may be because the pans are unequal in mass, or the arms in length, or it may be that the balance case is not level so that the stem which carries the beam is not vertical. Test for this by a spirit level, assuming the maker has set the stem at right angles to the bottom of the case so that the knife-edges and the plate on which the fulcrum rests will be horizontal when the bottom of the case is.

If the pointer does not come to zero[1], load one of the scale-pans with some shot or bits of lead foil until it does.

Then, since the beam is horizontal, we have if S and S' now denote the weights of the scale-pans and loads used to bring the beam horizontal,

$$Sa = S'b.$$

Let W be the weight of the body whose accurate weight is required.

Place it in the left-hand scale-pan and place weights P in the right-hand scale-pan until the beam is again horizontal.

Then $$(W + S)\,a = (P + S')\,b.$$

Hence $$Wa = Pb.$$

Now interchange W and P: if the beam remain horizontal the arms are equal, if not, let Q be the weights in the left-hand scale-pan which are required to balance W when in the right-hand scale-pan.

Then $$(Q + S)\,a = (W + S')\,b.$$

But $$Sa = S'b.$$

Hence $$Qa = Wb \qquad \text{(i)}$$

And we have already seen that

$$Wa = Pb \qquad \text{(ii)}.$$

[1] It is not important that the pointer should read zero exactly, provided its position with the balance unloaded is noted, and that it is always brought back to this position when a body is being weighed. To secure this it is usually simplest to adjust the balance so that the unloaded reading may be at the middle of the scale or very close to it.

Hence $$QWa^2 = PWb^2,$$

or $$\frac{a^2}{b^2} = \frac{P}{Q}.$$

Therefore $$\frac{a}{b} = \sqrt{\frac{P}{Q}} \quad\quad\quad \text{(iii)}.$$

Also dividing (i) by (ii)

$$\frac{Q}{W} = \frac{W}{P}.$$

Hence $$W^2 = PQ,$$

$$W = \sqrt{PQ}.$$

Thus P and Q are known weights: hence the ratio of the arms and the true value of W is found.

EXPERIMENT 9. *To determine the difference between the weights of the scale-pans.*

Level the balance as in Experiment 8, and note the position of the pointer on the scale. Interchange the scale-pans so that S may now hang from the arm b, S' from the arm a. If the pointer remains in the same position, the pans S and S' are equal in weight. If not, we can find their difference thus.

Let S be suspended from the arm a, and suppose that a small weight w must be added to S to make the beam horizontal.

Then $$(S+w)\,a = S'b.$$

Interchange S and S' and suppose now that w' in the pan S is required for equilibrium

$$S'a = (S+w')\,b.$$

Thus

$$S + w = S'\frac{b}{a},$$

$$S' = (S+w')\frac{b}{a}.$$

Hence

$$S - S' + w = (S'-S)\frac{b}{a} - w'\frac{b}{a}.$$

Thus

$$(S'-S)\left(1+\frac{b}{a}\right)=w+w'\,\frac{b}{a}.$$

Therefore

$$S'-S=\frac{w+w'\,\frac{b}{a}}{1+\frac{b}{a}}.$$

Hence if b/a is known we can find $S'-S$.

If w and w' be both very small and b/a very nearly unity, we shall not alter the value of this quantity much by putting b/a equal to 1, and then

$$S'-S'=\tfrac{1}{2}(w+w').$$

If b be accurately equal to a then clearly w is equal to w', but by proceeding as above and interchanging the pans we take into account the only important part of the effect due to a difference between b and a.

61. The common or Roman Steelyard. This balance consists of a lever AB, Fig. 135, supported on a knife-

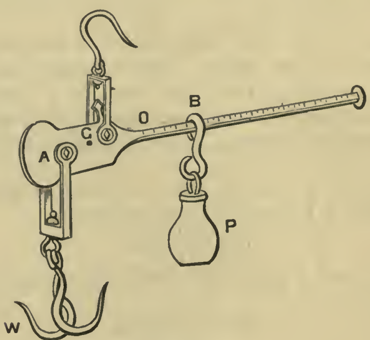


Fig. 135.

edge at C. A hook at A carries the pan in which the object to be weighed is supported. The arm BC has a number of divisions marked on it, and from these a standard weight P of constant magnitude can be suspended. The weight Q is determined by finding the division at which B must be sus-

pended in order that the beam may be horizontal. Practically then the steelyard is a balance such that the length of one arm can be adjusted: we weigh by adjusting this length, not by altering the weight. If the centre of gravity of the steelyard itself coincided with C the fulcrum, the weight W would be directly proportional to the distance from C of the point from which P is suspended, the beam could be graduated by dividing it into equal parts from C, each of these parts being equal to CA. In practice this is not the case: we have always to take into account the weight of the steelyard. This is done in the following way.

Suppose G is the centre of gravity and W the weight of the steelyard.

Let the scale-pan be unloaded and adjust P until the beam is horizontal. Let the position of P so found be O. The point G is usually between C and A, while O is between C and B. Then if a weight P were always kept at O it would just balance W at G. If then a weight Q be put into the scale-pan and another weight equal to P supported at some point along AC, the weight Q will be measured by the distance of this point from C. But the effect of the two weights P is the same as that of a single weight P placed at a distance beyond the second weight equal to CO. The weight Q is then measured by the distance of this weight from O, in other words the steelyard is graduated from O, not from C.

This may be put more briefly using symbols thus.

Let P suspended at B balance Q at A.

Then $$P \cdot CB = W \cdot CG + Q \cdot CA.$$

Now $$W \cdot CG = P \cdot CO \text{ by experiment.}$$

Hence $$P \cdot CB = P \cdot CO + Q \cdot CA.$$

Thus $$Q \cdot CA = P \cdot (CB - CO)$$
$$= P \cdot OB.$$

Hence $$Q = P\frac{OB}{CA}.$$

Hence if we take points 1, 2... etc., along AC such that their distances from O are AC, $2AC$... respectively, the weight Q is equal to P, $2P$, $3P$, etc., according as P is at 1, 2, 3, etc.

Thus the steelyard is graduated from O.

62. The Danish Steelyard. This consists of a bar AB, Fig. 136, terminating in a ball at B, the weight of which

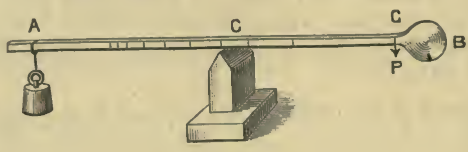


Fig. 136.

constitutes the power, the bar is graduated and the fulcrum is moveable. The body to be weighed is suspended from A, and the fulcrum C is shifted until the steelyard is horizontal. Then if P be the weight of the bar and G its centre of gravity, the moment about C of P acting at G is equal to that of W at A.

Thus to graduate the bar we have

$$W \,.\, AC = P \,.\, CG = P \,.\, (AG - AC).$$

Thus
$$(P + W)\, AC = P \,.\, AG,$$

or
$$AC = \frac{P \,.\, AG}{P + W}.$$

Thus by making W equal successively to P, $2P$, $3P$, etc., the successive graduations can be found.

63. The Letter Balance. (Roberval's Balance.) A common form of letter balance is shewn in Fig. 137.

The beam ACB turns about a fulcrum at C. The scale-pans are supported above the beam on knife-edges at A and B.

Two equal vertical rods AD, BE are attached below the scale-pans, and the lower ends D, E of these rods are connected

by joints to a horizontal bar DE parallel and equal to the

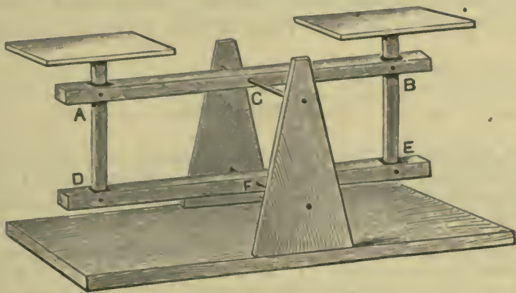


Fig. 137.

beam. This bar can turn about its middle point F, which is vertically below the fulcrum.

By this arrangement, as the balance swings, the rods AD, BE always remain vertical and the scale-pans horizontal.

Moreover it follows readily from the principle of work that if the arms of the beam are equal the weights may be placed anywhere on the scale-pans.

For if the beam be slightly displaced, every point of the one scale-pan rises the same vertical height h, say, while every point of the other falls an equal distance h. Hence if P and Q be the weights at any point of either scale-pan respectively, and there be equilibrium with the beam horizontal, we must have $P = Q$.

In an ordinary balance, if the weights be put on at any point of the scale-pan, the pan swings about its point of support until the centre of gravity of the weights is vertically under this point. The weights therefore always act vertically through the ends of the beam. If the pan be not free so to swing, then in general the arm at which the weights act, the horizontal distance that is between the fulcrum and the vertical line through their centre of gravity, will depend on the position of the weights in the scales and the balance if not

actually useless would be very troublesome to use. In the balance just described this difficulty is avoided by securing that each point of the pan rises or falls an equal amount.

EXAMPLES.

1. A balance has unequal arms. A piece of lead placed in the left pan weighs apparently 580 grams; when it is placed in the right pan its weight is apparently 560 grams. Calculate the ratio of the lengths of the arms of the balance.

2. A weightless rod $ABCD$ moveable about a fulcrum and twenty feet in length has weights P, $2P$, $3P$ and $4P$ attached to the rod at A, B, C and D which are at equal distances apart. If the rod be in equilibrium find the distance of the fulcrum from A.

3. In a given Roman steelyard where must the fulcrum be if the smallest weight that can be measured is half the moveable weight, assuming that the beam weighs 4 times the moveable weight?

4. Discuss the effect of an increase in the value of the acceleration of gravity (1) upon the sensitiveness, (2) upon the stability, of a balance.

5. Describe a balance and explain the conditions upon which the sensitiveness of a balance depends.

6. Why is it that the sensitiveness of a balance depends upon the sharpness of its knife-edges?

7. Describe and explain the various precautions which are necessary for the accurate determination of the mass of a body by means of a balance.

8. What is meant by stable, neutral, and unstable equilibrium? Give examples of each of these.

9. A false balance rests with the beam horizontal when unloaded, but the arms are not of equal lengths; a weight W, when hung at the end of the shorter arm b, appears to balance a weight P, and when hung at the end of the longer arm a it appears to balance a third weight Q; shew that

$$W=\sqrt{PQ}.$$

Can you suggest another way of ascertaining correctly the weight of W?

10. How would you compare the "stability" and the "sensibility" of two balances?

11. How may an Inspector with one standard pound test a tradesman's scales and weights?

12. An object is placed in one scale-pan of an ordinary balance and it is balanced by 20 lb. The object is then put into the other scale-pan, and now it takes 21 lb. to balance it. When both scale-pans are empty the scales balance. What is the matter with the balance, and what is the true weight of the object?

13. The arms of a false balance are in the ratio of 20 to 21. What will be the loss to a tradesman who places articles to be weighed at the end of the shorter arm if he is asked for 4 lb. of goods priced at 3*s.* per lb.?

14. The arms of a balance are 2 ft. long. One of the scale-pans is a circular disc, whose diameter is 6 inches, and which is fixed to the end of an arm of the balance by a rod passing through the centre of the pan and rigidly attached to the pan at right angles to its plane. Shew that a 1 lb. mass placed in such a pan may balance any mass between 18 and 14 ounces in the other pan.

15. A weight W is supported on a smooth inclined plane at an angle of 30° to the horizon, by a string attached to a point in the plane; find the tension of the string.

If in the preceding case the pressure on the plane is R, and in the case in which the string is inclined at an angle of 60° to the horizon [the inclination of the plane remaining the same] the pressure is R', prove that

$$2R = 3R'.$$

16. Find the horizontal force that would support a weight W on a smooth plane inclined at an angle of 60° to the horizon.

If on the same inclined plane the weight W is supported by two equal forces one acting horizontally and the other acting along the plane upwards, and the pressure on the plane is R, prove that $R = W$.

17. The height of an inclined plane is 4 feet and it requires a power P acting along the plane to support a weight W. If the height is altered, the length of the plane remaining the same, and $3P$, acting along the plane, is now necessary to support the weight W, find the new height.

18. A weight W is supported on a smooth plane inclined at an angle α to the horizon by means of a force inclined at an angle θ to the plane. Find the magnitude of the force, and the pressure on the plane.

If there is no pressure on the plane, in what direction does the force act?

19. A body of weight W is supported on a smooth plane inclined at an angle α to the horizon by means of a force inclined at an angle $\alpha + \beta$ to the horizon. Find the magnitude of this force.

If the force is vertical find the pressure on the plane.

20. A weight rests on a smooth inclined plane. Shew that the smallest force which will keep it in equilibrium must act along the plane.

If the weight be the weight of a ton, and the inclination of the plane be 45°, what is the power?

21. Two inclined planes of equal heights are so placed that they have a common vertex; a weight lies on each of the planes, and the weights are connected by a string which passes over the common vertex; in this position there is equilibrium; the lengths of the planes are respectively 6 and 3 feet; the weight which rests on the shorter plane is 10 lb. Find the other weight in one of the following ways, neglecting all friction:

(1) by means of the "triangle of forces,"

(2) by means of the "principle of work."

22. Find the ratio of the power to the weight on a smooth inclined plane when the former acts horizontally.

If the weight be the weight of a ton, and the inclination of the plane be 30°, what is the power?

23. A weight W is supported on a smooth plane inclined at an angle 30° to the horizon, by a string inclined at 60° to the horizon; find the tension of the string.

24. Find the horizontal force that would support a weight W on a smooth plane inclined at an angle of 45° to the horizon.

If on the same inclined plane the weight W is supported by two equal forces, one acting horizontally and the other acting along the plane upwards, find the pressure on the plane.

25. Find the ratio of the power to the weight when a body is kept in equilibrium on a smooth plane, inclined at an angle of 30° to the horizon, by a horizontal force.

A given force is applied to support a weight on an inclined plane. Will the greater weight be supported, when the force acts horizontally, and the plane is inclined at an angle of 30° to the horizon; or when the force acts parallel to the plane, and the plane is inclined to the horizon at an angle of 60°?

26. An inclined plane 14 feet long has one end 8 feet and the other end 10 feet above the level of the floor. If 294 ft.-lb. of work are done in dragging a mass of 1 cwt. up the plane, find the friction.

27. Describe the system of pulleys in which each string is attached to a bar from which the weight is suspended, and find an expression for its mechanical advantage, the weights of the pulleys being neglected.

If there are 3 strings attached to the bar, what power will support a weight of 35 lb.?

28. Describe the system of pulleys in which each pulley hangs in the loop of a separate string, and find an expression for its mechanical advantage, the weights of the pulleys being neglected.

If there are four moveable pulleys, what power will support a weight of 50 lb.?

29. Determine the relation of the power to the weight in a system of 4 moveable pulleys, of which the weight may be neglected, one end of each string being fixed to a beam. Find also how far the weight is raised when the power moves through 12 ft.

30. Find the mechanical advantage of the system of weightless pulleys in which each pulley hangs in the loop of a separate string, one end of which is fastened to a fixed beam; all the strings being parallel.

If there are 7 pulleys and the weight is 8 cwt. find the power in lb. wt.

31. Find the ratio of the power to the weight in the system of pulleys in which all the strings are parallel and are attached to the weight.

If there are 6 pulleys and the weight is 9 cwt. find the power in lb. wt.

32. In a certain system of pulleys it is found that the power descends 1 ft., while the weight rises 1 inch. What power will be required to raise a weight of 1 cwt.?

33. Describe the system of pulleys in which each string is attached to a bar from which the weight is suspended, and find an expression for its mechanical advantage, the weights of the pulleys being supposed equal.

If there are three strings attached to the bar, and the weight of each pulley is $1\frac{1}{2}$ lb., what power will support a weight of $42\frac{3}{4}$ lb.?

34. Describe the system of pulleys in which each pulley hangs in the loop of a separate string, and find an expression for its mechanical advantage, the weights of the pulleys being supposed equal.

If there are 4 moveable pulleys, each weighing $\frac{3}{4}$ of a lb., what power will support a weight of $56\frac{3}{4}$ lb.?

35. In the system of three equal pulleys, one fixed and two moveable, in which each string is attached to a bar supporting the weight W, prove that, neglecting the weights of the pulleys and the bar, if P is the power, $7P = W$, and find the point on the bar from which W must be suspended.

If, when there is equilibrium, one-third of the weight falls off, prove that the acceleration of the remainder of the weight will then be one twenty-third of the acceleration due to gravity.

36. Find the ratio of the power to the weight in that system of pulleys in which there is only one string.

In such a system a power P supports a weight W; if P and W are interchanged prove that the weight to be added to P to produce equilibrium is $\frac{W^2 - P^2}{P}$.

37. Draw carefully a system of pulleys in which each pulley hangs by a separate string and the ratio of the power to the weight is 1 to 32.

38. Find the conditions of equilibrium in the wheel and axle.

Shew that if the axle rest on rough bearings, the least power (acting downwards) that will raise a weight W is

$$\frac{b(1+\sin\lambda)}{a-b\sin\lambda}W;$$

where a, b are the radii of the wheel and axle and λ the angle of friction.

39. If in order to raise a weight of 144 lb. through an inch by means of 'a wheel and axle' I must move my hand through a distance of one foot, what power must I exert?

40. A balance is apparently in adjustment when no weights are in the scale-pans. A certain mass is put into the right pan and requires weights of 45·63 grams in the left to maintain equilibrium; on putting the mass into the left pan it is found that 45·81 grams are needed in the right. Find the mass and ratio of the arms.

41. In a wheel and axle the radius of the wheel is 3 feet. The axle is of square section, the side of the square being 6 inches long. Find (i) the greatest, (ii) the least vertical power that must be exerted to slowly lift a weight of 252 lb. in the usual manner.

42. A straight uniform lever AB, 12 feet long, balances about a point in it 5 feet from B, when weights 9 lb. and 13 lb. are suspended at A and B. Find the weight of the lever.

43. A straight uniform lever whose weight is 16 lb. balances about a point one foot from its middle point when weights 6 lb. and 10 lb. are suspended from its ends. Find the length of the lever.

44. Find the power required to support a weight W in a system of 4 pulleys in which each string is attached to the weight and the pulleys are supposed weightless.

If the weights of the pulleys are taken into account and each weighs 1 lb., find what power will support a weight of $78\frac{1}{2}$ lb.

45. Find the power required to support a weight W in a system of 4 pulleys in which each pulley is supported by a separate string, one end of which is fastened to a fixed beam, and the pulleys are supposed weightless.

If the weights of the pulleys are taken into account and each weighs 1 lb., find what power will support a weight of 65 lb.

46. Find the condition of equilibrium in the system of pulleys in which the same string goes round all the pulleys.

If a weight of 6 lb. just supports a weight of 28 lb., and a weight of 8 lb. just supports a weight of 42 lb.; find the number of pulleys, and the weight of the lower block.

47. Find the condition of equilibrium in the system of pulleys in which each string is attached to the weight.

If there are 5 moveable pulleys each weighing half-a-pound, and the weight is 35 lb., what is the power?

CHAPTER VII.

FRICTION.

64. Friction. The term "friction" has been used occasionally in the last chapter: we have seen that in many cases when a body rests on a surface the force between them is not wholly at right angles to the surface, but has a component along the surface. The surface is then said to be rough and the component of the force along the surface is called friction.

It remains now to consider the nature of friction a little more fully.

DEFINITION. *When a body is in contact with a rough surface and the impressed force has a component along the surface a force is called into play tending to balance this component and prevent motion. This force is called* **Friction.**

The **Direction of friction** is opposite to the component of the other forces resolved parallel to the surface, opposite that is to the direction in which motion would take place if there were no friction.

The **Amount of friction** up to a certain limit is always just sufficient to prevent motion, but only a limiting amount of friction can be called into play.

Thus if a body rest on a horizontal table the pressure of the table balances the weight, these forces are both vertical, there is no component in the direction of the surface and no friction is called into play. Apply a small force parallel to the surface, the body does not move, sufficient

friction is exerted just to stop the motion; increase the force still further, until the force parallel to the surface reaches a certain limit depending on the normal pressure and on the nature of the surface, the body does not move: when however this limit is exceeded, motion takes place.

Thus consider a ladder resting as in Example (1), p. 75, against a smooth wall on rough ground; when the foot of the ladder is near the wall the friction needed to maintain it in position is small; as the foot is withdrawn from the wall and the slope increases more friction is required and is called into play, until at last there comes a position in which the ladder begins to slip, the friction which needs to be exerted is greater than the ground can exert, the limiting position has been passed.

We may put this in symbols thus:

Let R be the normal pressure of the smooth wall, W the weight of the ladder acting vertically at its centre of gravity, Y the vertical stress at the foot and X the horizontal component of the action—the friction. Then, as in Example (1), p. 75,

Resolving vertically $\quad Y=W.$

Resolving horizontally $\quad X=R.$

Taking moments about the lower end

$$Rl\sin\alpha=\tfrac{1}{2}Wl\cos\alpha.$$

Hence $\quad X=\tfrac{1}{2}W\cot\alpha.$

Now as the angle α decreases this increases: when it becomes greater than the maximum friction which the ground can exert the ladder will slip. Let us call F this maximum friction, then X must be not greater than F.

Hence $\tfrac{1}{2}W\cot\alpha$ is not greater than F, and $\cot\alpha$ must not be greater than $2F/W$.

Many other observations shew us that there is a limit to the maximum amount of friction which can be called into play. The following experiments will help to shew on what the limiting amount depends.

Experiment 10. *To investigate the Laws of Limiting Friction.*

Thus take a well-planed board of hard wood and a small piece of wood, say six inches by nine in size; rub one surface of this small piece of wood as smooth as possible with sand-

paper and then cut the wood in two so as to have two pieces, one of about twice the size of the other, with surfaces as nearly alike as possible. Fix a screw eye into one end of each of these pieces of wood and attach a string to each. Fix a pulley at one end of the horizontal board over which the string may pass and support a scale-pan and weights. Secure some pieces of lead to the smaller piece of wood and thus make the weight of the two the same.

Place the larger piece of wood on the board and put a considerable weight, 5 or 6 kilos, on it. Pass the string from the wood over the pulley and suspend the scale-pan as in Fig. 138. Load the scale-pan until the wood just begins to

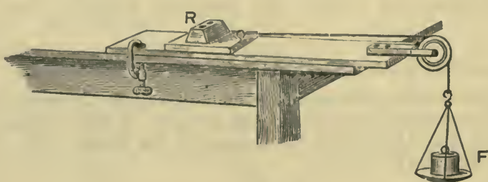


Fig. 138.

slip on the board. The weight required for this can be found fairly closely by gently tapping the board.

Note the total weight suspended; this measures the maximum friction which can be exerted between the board and the wood.

Note also the total weight supported by the board; including in this the weight of the wood itself[1]. This measures the normal stress between the two. Place more weights on the wood so as to increase the force between it and the board; it will be necessary to place more weights in the scale-pan in order to start the motion; the limiting friction is increased. Determine in this way the limiting friction for a number of loads and form a table in which the first column is the Limiting Friction, the second column the Normal Force between the surfaces.

[1] It is convenient to make this up to some fraction of a kilogram such as $\frac{1}{2}$ or $\frac{1}{4}$ by securing some pieces of lead to its upper side.

Divide the numbers in the first column by the corresponding numbers in the second; it will be found that the quotients are the same.

Thus when the surfaces in contact remain the same the ratio of the limiting friction to the normal force is a constant.

Now repeat the experiment, using the second or smaller piece of wood, it will be found that for the same normal stresses as previously the limiting friction is also practically the same. The areas of the surfaces in contact have been altered but not the nature of those surfaces or the state of their polish; the ratio of the limiting friction to the normal stress is unchanged.

Replace the wooden slide-piece by another of different material; the law, that the ratio of the limiting friction to the normal force is constant, still holds; but the value of this constant ratio differs from that found in the previous experiment.

65. Laws of Limiting Friction. We are thus led to the following laws of limiting friction.

(i) *The ratio of the limiting friction to the normal force between any two given surfaces is a constant.*

(ii) *This constant ratio depends on the material of the surfaces in contact and the state of their polish, but not on their area or shape.*

These two laws of limiting friction together with the definition and statements given in Section 64 are sometimes enunciated together as the Laws of Friction.

These laws, though they express fairly well the result of experiments, are probably not rigorously true; they are however usually employed in considering problems involving friction.

It is customary to give another law which however does not concern us in Statics.

When sliding motion takes place the ratio of the friction to the normal force is still found to be constant for any two given

surfaces and independent of the velocity; this constant however is slightly less than the constant ratio of the limiting friction to the normal force.

Thus it requires rather greater force to start a body moving on a rough surface against friction than to maintain it in motion with uniform speed when once started.

66. Coefficient of Friction. The constant ratio of the limiting friction to the normal stress for two surfaces of given material in a definite state of polish is called the Coefficient of Friction.

Thus if F is the limiting friction, R the normal force and μ the coefficient of friction, we have

$$\frac{F}{R}=\mu$$

or

$$F=\mu R.$$

The experiments described in Experiment 10 give one method of determining μ. The following method is generally more convenient.

Consider a body lying on a rough horizontal surface under its weight and the reaction of the surface, the body is in equilibrium and there is no friction. Tilt the horizontal surface, the weight will have a component down the surface. Friction is called into play to balance this component. Continue to tilt the surface until the body just begins to slide down; when this is the case the limiting amount of friction has been reached, and this limiting amount of friction is just equal to the component of the weight down the surface when tilted at the angle at which sliding just begins.

The following Experiment will enable us to verify the laws of friction and to find the coefficient of friction by this method.

EXPERIMENT 11. *To prove that on a rough surface the limiting amount of friction is proportional to the normal force, and to find the coefficient of friction.*

The apparatus consists of a mahogany board, Fig. 139, some 12 or 15 inches long and 3 or 4 inches in width. This is

hinged at one end to another similar board which can be clamped to the table. At the end of the second board remote

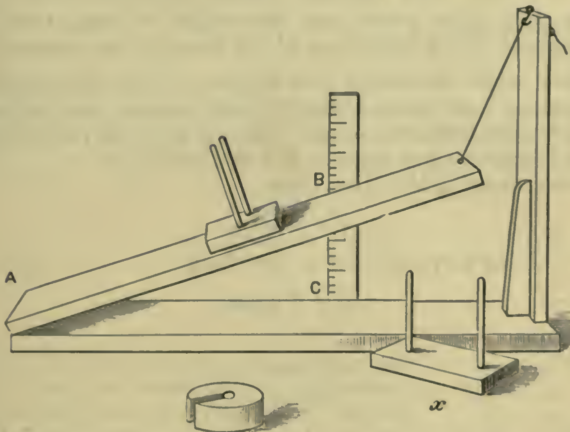


Fig. 139.

from the hinge a vertical support is fixed and the hinged board can be raised and secured in any position by means of a string passing through a small screw eye at the top of this support. A graduated vertical rod is also screwed as shewn at BC to the base board and the height of the plane at B can be easily measured on this rod. The point C is at some convenient distance (say 10 inches) from the hinge A: by dividing the height in inches by 10 we get at once the tangent of the inclination of the plane. Let the angle BAC be a.

One or more small boards of various sizes and materials can be placed on the inclined plane and weights can be placed on these small boards to vary the force between them and the inclined plane. In the apparatus shewn each small board has one or more thin brass rods screwed to it, the weights used in the other experiments can be piled on it so that each weight when the plane is tilted is prevented from slipping by the rod which passes through its centre.

Place one of the small boards with some convenient weight on it on the plane. Raise the plane, gently tapping it from time to time, until the small board begins to slide down. When this takes place note the height BC and thus find the tangent of the inclination of the plane to the horizon.

Now when the board just begins to slide the maximum friction has been reached and this just balances the component of the weight down the board. Thus if R be the normal force, F the friction in any position and W the weight,
we have resolving along the plane,

$$F = W \sin \alpha,$$

and resolving perpendicular to the plane

$$R = W \cos \alpha.$$

Hence

$$\frac{F}{R} = \frac{W \sin \alpha}{W \cos \alpha} = \tan \alpha.$$

Now experiment shews that, so long as the material and state of polish of the surfaces remain the same, the slipping just begins at the same angle. Thus replace the weights by others and repeat the experiment, the slipping takes place at the same angle as before. Replace the small board by another of the same material and polish but of different area; it will just begin to slip at the same angle as previously.

Hence if F now stand for the maximum amount of friction and α for the angle at which slipping takes place we see by the experiment that α is constant so long as the material and polish remain the same.

Hence since the ratio F/R is equal to $\tan \alpha$ we see that F/R is constant under these same conditions.

This ratio is defined to be μ the coefficient of friction.

Hence $$\mu = \frac{F}{R} = \tan \alpha.$$

Thus the coefficient of friction is found by reading BC the height of the plane and dividing it by AC the base.

67. Angle of Friction.

DEFINITION. *The angle α whose tangent gives the coefficient of friction is called the Angle of Friction.*

We can give a more general meaning to this angle thus.

Consider any body resting on a rough surface. Let the friction be F and the normal force R. The resultant of these two will be a force P, which combined with the other forces must maintain equilibrium. Now let P act at an angle θ with the normal force R. Then resolving P along and perpendicular to R, we have

$$P \cos \theta = R,$$
$$P \sin \theta = F.$$

Hence

$$\tan \theta = \frac{F}{R}.$$

Thus in any case the ratio F/R measures the tangent of the angle which the resultant force between the surface and the body makes with the normal to the surface.

Thus, when F reaches its limiting value, θ reaches its greatest value and becomes α the angle of friction. Hence the angle of friction is the angle which the resultant force makes with the normal when the friction has reached its greatest value; or in other words it is the greatest angle which the resultant force can make with the normal.

So long then as the resultant force is inclined to the normal at an angle less than the angle of friction, equilibrium is possible; when this angle becomes greater than the angle of friction motion must take place.

In the case of a body on an inclined plane the resultant force due to the plane must be always vertical, for the only other force is the weight; and the resultant of the normal force and the friction must just balance the weight. Again, the angle which the normal to the plane makes with the vertical is the angle of the plane. Thus the angle between the direction of the resultant force and the normal to the plane is the angle of the plane; so long as the angle of the plane is less than the angle of friction equilibrium is possible; when the angle of the plane is just greater than the angle of friction slipping begins.

The following is a table of approximate values for the coefficient of friction.

Wood upon wood	·5
" " " lubricated	·2
Wood upon polished metal	·6
" " " " lubricated ...	·12
Metal upon metal	·18
" " " lubricated	·12

EXAMPLES.

FRICTION.

1. Explain what is meant by the coefficient of friction. A cubical block rests on a rough plane, one end of which is gradually raised. Find the greatest value of the coefficient of friction which will just permit the block to slide down the plane before falling over.

2. State the laws of friction; and assuming that the friction is the same when a body is moving as when it is at rest, find the time taken by a body to fall down a rough inclined plane from rest.

3. A brick whose dimensions are $8 \times 4 \times 3$ inches rests on a rough plane in such a way that it cannot slip, and the plane is gradually tilted about a line parallel to one edge of the brick; shew that the angle through which the plane can be raised without upsetting the brick depends on which face of the brick is on the plane; find also the greatest and least angles for which the brick will just not upset.

4. What is meant by the angle of friction? The lower half of an inclined plane is rough, the upper half being smooth. A particle is allowed to slide from the top and is brought to rest by friction just as it reaches the bottom. Find the ratio of the friction to the weight of the particle, assuming it to be independent of the velocity.

5. A block W_1 rests on an inclined plane and is supported partly by friction and partly by the tension of a cord which passes over a pulley

at the top of the plane and carries a weight W_2. Shew how to find (graphically or otherwise) the values of W_2 which will (1) just prevent W_1 from slipping down, and (2) just make W_1 begin to slip up, when the coefficient of friction and the inclination of the plane are known.

6. A block of iron weighing 10 lb. rests on a level surface plate. A string attached to the block passes over a pulley so placed above the surface plate that the string makes an angle of 45° with the vertical. After passing over the pulley the string supports a weight. Find the least value of this weight which will make the block slip, the coefficient of friction being $\frac{1}{4}$th.

7. A heavy body is to be drawn up a rough inclined plane. If the force is the least possible prove that its inclination to the plane must equal the angle of friction.

8. Describe an experimental method for finding the coefficient of friction between two substances.

A semicircular disc rests in a vertical plane, with one part of its curved surface touching the ground and another touching a vertical wall. Shew that it will rest in any position in which its straight edge makes an angle with the vertical greater than

$$\cos^{-1}\frac{3\pi}{4}\frac{\mu+\mu^2}{1+\mu^2},$$

where μ is the coefficient of friction between the disc and the ground and between the disc and the wall. [The centre of gravity of a semicircular disc is at a distance from the centre equal to $(4/3\pi)$ times the radius.]

9. A heavy body rests on a rough plane inclined at the angle 30° to the horizontal, the coefficient of friction being $\frac{2}{\sqrt{3}}$.

A horizontal force along the plane is applied to the body, and is gradually increased until the body begins to move; find the direction in which the body begins to move, and the magnitude of the horizontal force.

10. A uniform rod AB of length $2a$ rests with the end B in contact with a rough vertical wall, and is supported by a smooth peg fixed at a distance b from the wall, and the end B is on the point of slipping upwards. Shew that the inclination of the rod to the vertical is then given by the equation $\sin^2\theta\,(\sin\theta-\mu\cos\theta)=\frac{b}{a}$, μ being the coefficient of friction between the rod and the wall.

11. How would you place a brick whose length is double its breadth, and breadth double its thickness, on a rough inclined plane, so as to be least likely to tumble over? Would it be less likely to slide down with one face in contact than another? Give reasons for your answers.

12. A weight W rests in equilibrium on a rough inclined plane, being just on the point of slipping down. On applying a force W parallel to the plane, the weight is just on the point of moving up. Find the angle of the plane and the coefficient of friction.

13. What is the coefficient of friction when a body weighing 50 lb. just rests on a plane inclined at 30° to the horizon? If the plane were horizontal what horizontal force would be required to move the body?

14. Find what horizontal force will be required to support a weight of 3 cwt. upon a smooth inclined plane, whose height is $\frac{3}{5}$ of its length.

15. A force P acting up an inclined plane supports a weight W on it. If R be the reaction of the plane, prove that

$$P : W : R :: \text{height of plane} : \text{length} : \text{base}.$$

16. A mass of 1 cwt. rests on a rough inclined plane of angle 30°. If the coefficient of friction be $1/\sqrt{3}$ find the greatest and least forces which, acting parallel to the plane in both cases, can just maintain the mass in equilibrium.

HYDROSTATICS

CHAPTER I.

STATES OF MATTER.

1. Solids and Fluids.

In Dynamics and Statics we have dealt with some parts of the Mechanics of Solids. A Solid Body such as a lump of iron or a lump of wood has a definite Volume. It also has a definite Shape. If force be applied to it, both the shape and the volume are generally changed, though the change in many cases is very small. The force required to produce a given change of shape or size differs for different substances.

Great force must be applied to a lump of iron to produce an appreciable alteration in shape or volume; a solid india-rubber ball can be squeezed from its spherical form by a much smaller force than is necessary in order to change to an equal extent the shape of a similar sphere of iron.

Again, if the force applied be not too large, both the iron and the india-rubber will regain their original shape and volume when it is removed. Iron and india-rubber are both Elastic Substances.

Definition. *An* Elastic *Substance is one which has a definite shape and volume when free from the action of external forces; when such forces are applied, the shape, or the volume, or both are changed; when the forces are removed—provided they have not been too great—the substance recovers its shape and its volume.*

If the forces are too great the body may be strained beyond recovery, the limits of its elasticity may be passed; when the forces are removed the body does not regain its former shape and volume.

A solid body then can offer resistance (*a*) to forces tending to change its volume, (*b*) to forces tending to change its shape. In consequence of the former it is said to have **Volume Elasticity**; in consequence of the latter it is said to have **Elasticity of Form** or, as it is called, **Rigidity.**

A perfectly rigid body is one in which no change of shape is produced by the action of a finite force; no known body is perfectly rigid, though the rigidity of most solids is so great that for many purposes we may treat it as perfect.

Solids possess these two kinds of elasticity in very different degrees. Thus the shape of a piece of india-rubber is easily altered; experiment however shews that it requires considerable force to change its volume. A piece of cork can by the application of moderate force be squeezed into a much smaller volume; its shape however is not necessarily greatly changed in the process. In comparison with its volume elasticity the rigidity of cork is considerable.

Any body which has **Elasticity of Form** *or* **Rigidity** *is called a* **Solid.** *In consequence of its rigidity the body preserves its shape.*

It is a matter of everyday experience that there are numbers of bodies which exhibit little or no tendency to retain their shape. Ice is a solid, on melting it becomes water, the particles of the water slide freely over each other and the shape assumed depends on that of the vessel in which it is contained. Water, alcohol and numerous other substances can be poured from one vessel to another. Such substances have practically no rigidity; they offer practically no resistance to forces tending to produce sliding motion of their particles and thus to change their shape. They are called **Fluids.** We distinguish then between **Solids** and **Fluids** thus.

DEFINITIONS. *A* **Solid** *is a body which can offer permanent resistance to forces tending to change its shape.*

A **Fluid** *is a body which can offer no permanent resistance to forces tending to change its shape.*

In other words, a Solid has rigidity, a Fluid[1] has no rigidity.

2. Fluids. Water and alcohol have been instanced as examples of fluid bodies, we can readily pour them from one vessel to another, and these fluids adapt themselves almost instantaneously to the shape of the vessel into which they are poured; they offer practically no resistance to forces tending to change their shape; moreover the change of shape follows very rapidly on the application of the slightest force.

There are other substances, however, in the case of which, time is necessary, before a change of shape will occur under the action of a force. Honey or treacle can be poured from one vessel to another, but they pour slowly. We can make a heap of treacle in the middle of a dish or plate, but on leaving it, the heap is gradually flattened out and the plate covered. The treacle has no permanent rigidity, it can offer no resistance to a small force, such as its weight, if that force acts for a sufficient time; treacle like water is a fluid, but it has the property of **Viscosity** and in consequence yields slowly to forces tending to change its shape. A **Perfect Fluid** is one which yields instantaneously to such forces. No fluid in nature is perfect, water and alcohol have some slight viscosity, but the amount is so slight that they may be treated as perfect in comparison with fluids such as treacle, honey or glycerine, which are called **Viscous Fluids.**

It is sometimes difficult to draw the line between a solid and a fluid or between a viscous fluid and one which is practically perfect.

Thus consider a stiff jelly made by melting gelatine in water and allowing it to solidify; the jelly retains its shape when cold and recovers it again after being slightly squeezed, it is a solid—on mixing it with more water we obtain a sticky, viscous liquid, if the quantity of water be considerably increased the viscosity can be made very small indeed and the fluid is practically perfect.

[1] It will not however be sufficient to give this last statement as a definition of a fluid unless at the same time a definition of rigidity be given.

Again, a piece of pitch or of cobblers' wax has a definite shape, but it can only retain it for a short time, if placed on a flat surface it will gradually flow over the whole; pitch then must be classed as a fluid, but as a very viscous one. We may shew this by placing some pitch in a wide-necked funnel and leaving it in a fairly warm place, the pitch will gradually flow through the funnel. The same fact is illustrated by supporting a rod of sealing-wax at two points near its ends respectively, in time the rod is seen to bend, sinking in the middle; a small force produces change of shape but time is necessary in order that the effect may take place.

We must also distinguish between viscous fluids and plastic solids. Beeswax and paraffin wax are both solids, they have a definite shape and will retain it indefinitely, the application of quite a small force however is sufficient to mould a piece of beeswax into a new form; the rigidity of such a substance is extremely small, the limits within which it will recover its form are very narrow; it is said to be Plastic. If a paraffin candle be substituted for the sealing-wax in the experiment just described, it will not sag as the wax did, it can support its weight without continuous yielding and does not gradually change in shape under so small a force. The paraffin is a soft solid.

We thus see the importance of the word *permanent* in the above definition of a fluid. In the experiments to be described it will generally be assumed that the fluids employed are not viscous, though we shall find that in dealing with the equilibrium of fluids—**Hydrostatics**—we need not consider viscosity. See Section 15.

3. Liquids and Gases.

Fluids then differ from **Solids** in that they have *no* **Rigidity**, they have however **Volume Elasticity.** A fluid, like a solid, will resist a force tending to reduce its volume, and will recover that volume when the force is withdrawn.

But Fluids can be divided into two main classes possessing this property in very different degrees.

Water and air are both fluids ; very great force is needed to produce even a small change in the volume of a mass of water, that of a mass of air (Section 79) can be changed easily.

Some fluids are practically **Incompressible**, the change of volume produced by the application of even a very large force is extremely small. Such fluids are called **Liquids.**

Water, Oil, Alcohol, Vinegar are liquids. Other fluids are very easily compressible; these are called **Gases.** Such are Air, Oxygen, Hydrogen, Carbonic Acid.

DEFINITIONS. *A* **Liquid** *is a substance which can offer no permanent resistance to forces tending to change its shape, but which offers very great resistance to forces tending to diminish its volume.*

A **Gas** *is a substance which can offer no permanent resistance to forces tending to change its shape, and which offers only a small resistance to forces tending to diminish its volume.*

To illustrate the difference between a Liquid and a Gas let us imagine a cylinder closed with a tightly-fitting piston and suppose the area of the piston to be 100 square centimetres. Let there be a litre (1000 c. cm.) of water in the cylinder, then the depth of the water will be 10 cm.

Now suppose a weight of 100 kilogrammes is placed on the piston so that each square centimetre of the piston has to carry an additional weight of 1 kilogramme, then it has been shewn that—supposing the piston to move without friction—it would sink by about one two-thousandth ($\frac{1}{2000}$) of a centimetre; the change of volume of the whole litre produced by this pressure would be $\frac{1}{20}$ of a cubic centimetre, the change in each cubic centimetre therefore would be $\frac{1}{20000}$ of a cubic centimetre.

Thus the volume of a given mass of a liquid is very nearly constant and is only changed very slightly by the application of considerable forces. We may treat a liquid as a fluid of invariable density. See Section 5.

Now let us suppose that the water is removed from the cylinder and replaced by an equal volume of air at atmospheric pressure[1]. Then, on repeating the experiment, it would be found that the piston—if it were perfectly frictionless—would sink about five centimetres, the volume of the air would be about halved. Each cubic centimetre would now occupy half a cubic centimetre. Air at atmospheric pressure is about 10,000 times as compressible as water.

[1] See Section 67.

The density of the air is hereby doubled. Gases then are fluids the density of which depends on the pressure to which they are subjected.

The experiment could not be carried out in this simple form because of the friction of the piston against the sides of the cylinder, but this difficulty can be avoided by means of a suitable modification of the apparatus. See Section 79.

4. Free Surface of Liquids.

There is moreover another distinction between a liquid and a gas. Imagine that the walls of the cylinder just described are continued some distance above the piston, on raising the piston the level of the upper surface of the water still remains at about 10 centimetres from the bottom, above it there will be an almost empty space containing a little water vapour; the water will have a free surface separating it from the empty space above.

If, however, the cylinder contain a gas this will no longer be the case, the gas will expand as the piston rises, the whole space below the piston will be occupied by the gas, its density and the pressure it exerts on the sides of the cylinder will diminish; there will be no free surface.

Thus we may say that an incompressible fluid—a **Liquid**—is a substance which can offer no permanent resistance to forces tending to change its shape, but which has a definite density; it will therefore not increase indefinitely in volume if the force on its surface be diminished; when placed in any vessel it will occupy the lower portion of the vessel completely and will have a free surface.

A **Gas** is a substance which can offer no permanent resistance to forces tending to change its shape; its density however depends on the forces impressed on its surface, it will increase indefinitely in volume if these forces be sufficiently diminished, and, if placed in an empty closed vessel of any size, will fill it completely and have no free surface.

For the purposes of this book we may treat liquids as incompressible.

5. Density. Equal volumes of different substances differ in mass and therefore also in weight. We have already (*Dynamics*, Section 12) given a definition of the term **Density** which has been used in the last Section, and deduced some results from it.

The definition is as follows:

Definition. *The* **Density** *of any homogeneous substance is the mass of unit volume of that substance.*

It follows from this definition that to determine the density of a body we must find the number of units of mass in the unit of volume, we require therefore to know the unit of mass and the unit of volume; if these be the gramme and the cubic centimetre respectively we may say that the density is so many grammes per cubic centimetre. Thus in these units the density of water is 1 gramme per c.cm., that of iron 7·76 grammes per c.cm. In any other units the numerical measures of the densities of these substances would generally be different. Thus a cubic foot of water contains 998·8 oz. or 62·321 lbs.; hence the density of water is 998·8 oz. per cubic foot or 62·321 lbs. per cubic foot; iron is 7·76 times as dense as water, hence its density is $7{\cdot}76 \times 62{\cdot}321$ lbs. per cubic foot.

From the above definition of density we can find a relation between the Mass, Volume, and Density of a body.

Proposition 1. *To shew that if the mass of a homogeneous body be M grammes, its density ρ grammes per cubic centimetre and its volume V cubic centimetres, then $M = V\rho$.*

For by the definition,

the mass of 1 c.cm. $= \rho$ grammes,

therefore the mass of 2 c.cm. $= 2\rho$ grammes,

the mass of 3 c.cm. $= 3\rho$ grammes,

hence the mass of V c.cm. $= V\rho$ grammes.

Therefore $$M = V\rho.$$

We may write this as

$$\rho = \frac{M}{V},$$

and thus we have the result that the density of a homogeneous substance is the ratio of its mass to its volume.

A result similar to the above holds for any other consistent system of units.

To determine then the density of a piece of homogeneous material we require to know its mass, which is obtained in terms of a standard mass by weighing (*Statics*, Sections 59, 60), and its volume, which may be found in some cases by direct measurement, in others by the displacement method (*Dynamics*, Experiment 4), in others again by one or other of the methods described in the following pages, Experiments 15 etc. In any case we should notice that the measure of the density will depend on the units adopted for the measurement of the mass and the volume; these units must be known in order to determine the density completely.

6. Specific Gravity. In many cases, however, we are only concerned with the relative masses or the relative weights of equal volumes of two substances; it may be sufficient for us to know that a lump of iron is 7·76 times as heavy as an equal volume of water, or that the weight of a piece of pine wood is about ·56 of that of an equal volume of water.

DEFINITION. *The* **Specific Gravity** *of a substance is a number which expresses the ratio between the weight of the substance and that of an equal volume of some standard substance, usually water.*

Thus, if W be the weight of this substance, W' the weight of an equal volume of some standard substance, and σ the specific gravity, then we have

$$\sigma = \frac{W}{W'}.$$

We notice in the first place that the specific gravity of a body, being the ratio of two weights, is a number and does not depend on the units in which the weights are expressed, so long of course as those units are the same for the two. The specific gravity merely expresses the number of times which the weight of a certain volume of some standard is contained in

the weight of an equal volume of the substance. A cubic inch of iron is 7·76 times as heavy as a cubic inch of water whether it be measured in grammes weight, in pounds weight or in any other units.

We can also find a relation between the weight, the volume and the specific gravity of a substance thus.

PROPOSITION 2. *To shew that if the weight of a homogeneous body be W, its volume V, and its specific gravity σ, then $W = V\sigma\omega$, where ω represents the weight of unit of volume of the standard substance.*

Let W' be the weight of an equal volume V of the standard substance, then, since ω is the weight of each unit of volume of the standard, the weight of V units of volume is $V\omega$.

Hence W' is equal to $V\omega$.

But $$\sigma = \frac{W}{W'} = \frac{W}{V\omega}.$$

Therefore $$W = V\sigma\omega.$$

Thus if we know the volume of a body, its specific gravity referred to some standard substance, and the weight of unit of volume of that standard, we can calculate the weight of the body.

The calculation is simplified if we take water as the standard substance, the volume of 1 cubic centimetre as the unit of volume and the weight of 1 gramme as the unit of weight, for, since the weight of 1 cubic centimetre of water is 1 gramme weight, we have in this case the weight of the unit of volume as the unit of weight; thus

$$\omega \text{ is unity and } W = V\sigma.$$

Hence, *The weight of a body in grammes weight is found by multiplying its specific gravity by its volume in cubic centimetres.*

This same simple relation does not in general hold for other systems of units; thus if the weight of a pound be the unit of weight and one cubic foot the unit of volume, since a cubic foot of water has 62·321 pounds weight, the value of ω is 62·321 pounds weight, so that in order to find the weight of a body in pounds weight we multiply its specific gravity by its volume in cubic feet and by 62·321.

7. Definitions of Specific Gravity.

We can put the definition of specific gravity into various other forms which may be useful.

Thus, let W be the weight of a body, M its mass, V its volume and ρ its density.

Let σ be its specific gravity referred to some standard substance; consider an equal volume V of this standard, let W' be its weight, M' its mass and ρ' its density.

Then we have
$$W = Mg = \rho Vg,$$
$$W' = M'g = \rho' Vg.$$

Hence
$$\sigma = \frac{W}{W'} = \frac{Mg}{M'g} = \frac{M}{M'}.$$

Thus, *The specific gravity of a body is the ratio of the mass of the body to the mass of an equal volume of some standard substance.*

Again, since
$$M = \rho V,\ M' = \rho' V,$$
we have
$$\sigma = \frac{W}{W'} = \frac{M}{M'} = \frac{\rho V}{\rho' V} = \frac{\rho}{\rho'}.$$

Thus, *The specific gravity of a body is the ratio of its density to the density of the standard substance.*

Specific gravity is therefore sometimes spoken of as relative density.

Again, since $\sigma = \rho/\rho'$, we have $\rho = \sigma\rho'$.

Thus, *We can find the density of a body by multiplying its specific gravity by the density of the standard substance.*

Now, on the C.G.S. system, the density of water is 1 gramme per cubic centimetre, the value of ρ' then is 1 gramme per cubic centimetre, and

$$\rho = \sigma \text{ grammes per cubic centimetre.}$$

Thus, *On the* C.G.S. *system the numbers expressing the density and the specific gravity of a body are the same.*

It does not, of course, follow that the density and the specific gravity are the same, and the distinction between them must be carefully borne in mind.

8. The Standard Substance.

When determining the specific gravities of solids and liquids water is usually adopted as the standard. It is suitable for the purpose for it can readily be obtained in a state of purity.

The density of water like that of other substances depends on the temperature; thus the mass of a given volume of water is not always the same. Water above 4° C. expands as the temperature is raised; thus a volume of 1 cubic centimetre will weigh less when warm than it does when cold. In order to be quite accurate it is necessary to specify the temperature of the standard substance; for water this temperature is taken at 4° C., for at this temperature it is found (see Glazebrook, *Heat*, §§ 88—90) that water is denser than at any other. A given volume will weigh more at 4° C. than at any other temperature.

The variation of density, however, due to change of temperature is very small and for the purposes of this book need not be taken into account; we shall assume therefore as the weight of 1 cubic centimetre of water 1 gramme weight and as the weight of 1 cubic foot of water 62·321 pounds weight at any temperature.

The densities of gases are excessively small when compared with those of most solids and liquids. Thus the density of hydrogen at 0° C. and 760 mm. of pressure is ·0000896 grammes per c.cm., that of oxygen is sixteen times as great; hence, if water were taken as the standard substance in experiments on gases, the values of the specific gravities would all be small fractions; to avoid this it is usual to adopt hydrogen at a standard pressure and temperature as the standard substance.

9. Measurement of Specific Gravity.

To determine the specific gravity of a body we need to find its weight and the weight of an equal volume of some standard substance—water. In some cases this can be done directly, some experiments illustrating this are given below; in most cases, however, the methods described in Chapter VI. must be followed.

Experiment 1. *To determine directly the specific gravities of various substances.*

Make a number of cubes of the various substances wood, lead, iron, brass, copper, etc., all of the same size. Make a cubical box of some material such as brass into which the cubes will exactly fit. Weigh each of the cubes carefully and let the weights be W_1, W_2, etc.; weigh the box empty, then fill it with water and weigh again, the difference gives the weight of water filling the box, let it be W'; the volume of this weight of water is equal to that of each of the cubes. Divide the weights W_1, W_2, etc. by W', the respective quotients will give the specific gravities required.

The numbers thus found will also give the densities of the cubes in grammes per cubic centimetre. These however may be determined directly thus.

Experiment 2. *To determine the density of a cubical block.*

Determine by the balance the mass of the block in grammes. Measure with the calipers or the screw gauge (*Dynamics*, Section 7) the length of an edge and by cubing this find the volume in cubic centimetres.

Divide the mass in grammes by the volume in cubic centimetres, the quotient is the density in grammes per c.cm. Since the block may not be quite cubical it is best, in order to determine the volume, to measure the length of each of the three edges which meet at one angular point and multiply these three together.

Experiment 3. *To verify that the mass of* 1 *cubic centimetre of water is* 1 *gramme.*

Obtain a hollow vessel the volume of which can be found by measurement; for this purpose a hollow cylindrical vessel is convenient. Measure with the caliper-compasses or otherwise the interior diameter of the cylinder in centimetres, and, dividing this by 2, find the radius, r centimetres. Measure also the depth, d centimetres, of the cylinder. Then the area of the bottom is πr^2 square centimetres and the volume of the cylinder is $\pi r^2 d$ cubic centimetres. Weigh the cylinder empty, fill it with water and weigh again and thus obtain the mass of water filling the cylinder; let it be W grammes.

Then it will be found that

$$W = \pi r^2 d.$$

Hence the mass in grammes is numerically equal to the volume in cubic centimetres, thus the mass of 1 cubic centimetre of water is 1 gramme.

Instead of weighing the cylindrical vessel and its contents it may be more convenient to weigh the water in a small flask or bottle. Pour from the flask sufficient water to fill the cylinder and then weigh again, the difference will give the mass of water required to fill the cylinder.

EXPERIMENT 4. *To determine the specific gravity of a fluid.*

Obtain a flask or bottle with a narrow neck—one holding about 50 c.cm. will be suitable—make a scratch on the neck with a fine file. Weigh the flask empty, let its weight be W, fill it with water up to the scratch, dry the outside and weigh again. Let the weight be W_1. We thus obtain the weight $W_1 - W$ of a volume of water filling the flask up to the mark. Empty the water and fill the flask up to the mark with the liquid to be examined. Weigh again, let the weight be W_2. Then the weight of liquid which fills the flask to the mark is $W_2 - W$, and the weight of an equal volume of water is $W_1 - W$.

Thus the specific gravity[1] of the liquid is given by

$$\sigma = \frac{W_2 - W}{W_1 - W}.$$

There are numerous other methods of finding specific gravity. An account of these and of the precautions required in the use of the specific gravity bottle will be given in Chapter VI.

The following examples illustrate this part of the subject.

Examples. (1) *A sphere* 10 *cm. in radius has a mass of* 5 *kilogrammes, find its density.*

The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

Thus the volume of the given sphere is

$$\frac{4}{3} \cdot \frac{22}{7} \cdot 1000 \text{ c. c.}$$

[1] This method is usually known as that of the specific gravity bottle. For further details see Section 61.

Its mass is 5×1000 grammes.

Hence its density is

$$\frac{5 \,.\, 3 \,.\, 7}{4 \,.\, 22} \text{ grammes per c. cm.,}$$

and this reduces to 1·194 grammes per c.cm.

(**2**) *The specific gravity of iron is* 7·76, *find the weight of* 1000 *cubic feet of iron.*

The weight of 1 cubic foot of water is 62·32 lbs. weight.

Thus the weight of 1000 cubic feet of water is 62321 lbs. weight, and that of 1000 cubic feet of iron is

$$62321 \times 7{\cdot}76,$$

or 483611 lbs. weight.

(**3**) *The specific gravity of glass is* 2·5. *What volume of glass weighs* 1 *cwt.?*

A cubic foot of water weighs 62·321 lbs. weight.

Thus the volume of 1 lb. weight of water is 1/62·321 c. feet and the volume of 1 cwt. or 112 lbs. is 112/62·32 c. feet.

Now a given volume of glass is 2·5 times as heavy as the same volume of water.

Hence the volume of a given mass of glass is 1/2·5 of that of an equal mass of water.

Thus the volume of 1 cwt. of glass is

$$\frac{112}{2{\cdot}5 \times 62{\cdot}32} \text{ c. feet,}$$

and this reduces to ·7189 c. feet.

(**4**) *The mass of* 5 *cubic feet of ebony is* 365 *lbs., find its density in grammes per cubic centimetre.*

The density of ebony in lbs. per cubic foot is 365/5 or 73.

Now 1 lb. contains 453·6 grammes, 1 c. foot contains 28315 c. cm.

Thus the mass of 28315 c. cm. of ebony is $73 \times 453{\cdot}6$ grammes.

Hence the density is

$$\frac{73 \times 453{\cdot}6}{28315},$$

or 1·168 grammes per c. cm.

10. Values of Specific Gravities.

The following is a Table of Specific Gravities of some few substances.

SOLIDS.

Aluminium	2·7	Iron	7·76
Amber	1·1	Lead	11·4
Beech wood	·69—·8	Marble	2·7
Brass	8	Oak wood	·74
Bronze coinage	8·66	Pine wood	·56
Copper	8·95	Silver	10·57
Cork	·24	Slate	2·1
Diamond	3·5	Zinc	7·2
Gold	19·3	Wax (bees)	·96
Glass	2·5—3·6	Ice	·918.
Tin	7·29		

LIQUIDS.

Alcohol	·795	Nitric acid	1·50
Carbon disulphide	1·28	Mercury	13·6
Chloroform	1·53	Olive oil	·915
Glycerine	1·26	Petroleum	·84—·878
Sulphuric acid	1·85	Sea water	1·026.

*11. Propositions on Density and Specific Gravity.

There are two other Propositions connected with this part of the subject which may be useful.

*PROPOSITION 3. *To find the mass and the density of a mixture of any number of substances whose volumes and densities are known.*

Let V_1, V_2, etc., be the volumes of the substances, ρ_1, ρ_2, etc. their densities. Let us suppose that there is no chemical action between the substances on mixing, then the volume of the mixture is the sum of the volumes of the component parts. Let it be V and let ρ be the density of the mixture which we suppose to be homogeneous.

Then, since the volume is unchanged by mixing,

$$V = V_1 + V_2 + V_3 + \text{etc.},$$

and since the mass of the mixture is the sum of the masses of its components,

$$V\rho = V_1\rho_1 + V_2\rho_2 + V_3\rho_3 + \text{etc.}$$

Thus
$$\rho = \frac{V_1\rho_1 + V_2\rho_2 + \dots}{V_1 + V_2 + \dots}.$$

If the volume change on mixing and become V' the first relation above given will not be true, but we shall have

$$V'\rho = V_1\rho_1 + V_2\rho_2 + V_3\rho_3 + \dots.$$

We can establish a similar formula, using specific gravity and weight instead of density and mass.

Thus if σ_1, σ_2,... be the specific gravities and ω the weight of a unit of volume of the standard substance, we have, assuming no chemical action to occur,

$$V = V_1 + V_2 + \dots + \text{etc.}$$

$V\sigma\omega$ = weight of whole = sum of weights of components

$$= V_1\sigma_1\omega + V_2\sigma_2\omega + \dots.$$

Thus, dividing by ω,

$$V\sigma = V_1\sigma_1 + V_2\sigma_2 + \dots + \text{etc.}$$

*PROPOSITION 4. *To find the volume and density of a mixture of substances whose masses and densities are known.*

Let the masses be M_1, M_2, etc. and the densities ρ_1, ρ_2, etc. Let M be the mass of the mixture, ρ its density.

Then the volumes of the separate substances are respectively M_1/ρ_1, M_2/ρ_2,... and the volume of the mixture is M/ρ.

Thus we have

$$M = M_1 + M_2 + M_3 + \dots \text{etc.}$$

$$\frac{M}{\rho} = \frac{M_1}{\rho_1} + \frac{M_2}{\rho_2} + \frac{M_3}{\rho_3} + \dots \text{etc.}$$

Hence
$$\rho = \frac{M_1 + M_2 + \dots \text{etc.}}{\dfrac{M_1}{\rho_1} + \dfrac{M_2}{\rho_2} + \dots \text{etc.}}$$

A similar formula connects together the weight and specific gravity of a mixture, for we have, if W_1, W_2,... etc. be the weights, σ_1, σ_2,... etc. the specific gravities of its components,

$$W = W_1 + W_2 + W_3 + \dots + \text{etc.},$$

$$\frac{W}{\sigma\omega} = \frac{W_1}{\sigma_1\omega} + \frac{W_2}{\sigma_2\omega} + \frac{W_3}{\sigma_3\omega} + \dots + \text{etc.}$$

Examples. (**1**) *If a volume of* 10 *c.cm. of a liquid of density* ·8 *grammes per c.cm. be mixed with* 15 *c.cm. of a liquid of density* ·7 *grammes per c.cm., find the density of the mixture.*

The volume of the mixture is $10+15$ or 25 cubic centimetres.

The mass of the mixture is $10\times ·8+15\times ·7$ or 18·5 grammes.

Hence its density is 18·5/25 or ·74 grammes per c.cm.

(**2**) *An alloy of zinc* (*sp. gr.* 7·2) *and copper* (*sp. gr.* 8·95) *has a mass of* 467 *grammes. Its volume is* 60 *c.cm. Find the volume of each component.*

Let v_1 c.cm. be the volume of the zinc and v_2 c.cm. that of the copper.

Then the mass of the zinc is $7·2\times v_1$ grammes, that of the copper is $8·95\times v_2$ grammes. The sum of these two is the total mass 467 grammes, the sum of the two volumes is the total volume 60 cubic centimetres.

Thus $$v_1+v_2=60,$$

$$7·2\, v_1+8·95\, v_2=467.$$

Hence, solving these equations,

$$1·75\, v_1=537-467=70,$$

$$1·75\, v_2=467-432=35.$$

Thus $$v_1=40 \text{ c.cm.}$$

$$v_2=20 \text{ c.cm.}$$

(**3**) *A mixture is made of* 14 *cubic centimetres of sulphuric acid* (*specific gravity* 1·85) *and* 6 *cubic centimetres of water. The specific gravity of the mixture is found to be* 1·615. *Determine the amount of contraction which has taken place.*

If there were no contraction the volume of the mixture would be $14+6$ or 20 c.cm.; let the actual volume be V c.cm. Then, since the density is 1·615 grammes per c.cm., the mass is $V\times 1·615$ grammes.

The masses of the components are $14\times 1·85$ or 25·9 grammes and 6 grammes respectively.

The mass of the mixture is the sum of the masses of its components.

Hence $$V\times 1·615=25·9+6=31·9.$$

Therefore $$V=19·75 \text{ c.cm.}$$

Hence the contraction required is $20-19·75$ or ·25 c.cm.

EXAMPLES.

1. What is meant by the density of a substance?

How would you find the density of water?

2. The density of a substance being defined as the mass of a unit of volume of the substance, shew precisely how the density of a liquid may be experimentally determined.

3. Describe the experiments you would make in order to determine the mass of a cubic centimetre of water.

4. Explain clearly the distinction between specific gravity and density and shew how the numerical value of these quantities depends on the choice of fundamental units.

5. Find the density and specific gravity of the following body:

A rectangular pillar having a square base each side of which is one foot in length, the height of the pillar being 10 feet and its weight half a ton.

(The weight of a cubic foot of water may be taken as 1000 ounces.)

6. The density of copper is 8·95 grammes per c.cm. The diameter of a piece of copper wire is 1·25 mm. and its length 1025 cm.; find its mass.

7. Find the density of a cylinder 1 foot in height and 6 inches in radius whose mass is 60 lbs.

8. Find the density of a sphere 10 cm. in radius and 5 kilogrammes in mass.

9. Determine the density of the cylinder described in Question 7 in grammes per c.cm.

10. Find the density of a pyramid on a triangular base each side of which is 10 cm. and which has an altitude of 30 cm., the mass of the pyramid being 8 kilogrammes.

11. The density of mercury is 13·59 grammes per c.cm.; find it in grains per cubic inch.

12. Compare the densities of a sphere 5 cm. in radius, 5 kilos. in mass, and of a cylinder 1 foot in height, 6 inches in radius and 60 lbs. in mass.

13. Find the mass in lbs. of a cube of gold each side of which is 4 inches.

14. An iceberg is 30 fathoms high, 40 fathoms wide and 30 fathoms thick; find its mass in tons.

15. A carboy of sulphuric acid has a mass of 98 kilogrammes; find its volume.

16. It is desired to float a piece of slate a metre square by 5 centimetres thick by means of cork floats. What volume of cork is required?

17. Find the specific gravity of a mixture of glycerine and alcohol (i) in equal parts by weight, (ii) in equal parts by volume.

18. A piece of brass is made from 2 lbs. of copper and 3 lbs. of zinc, the volume of the copper being 6 cubic inches, and that of the zinc $13\frac{1}{2}$ cubic inches; find the specific gravity of the brass.

19. A cylindrical tube, 16 cms. long, holds when full 1 gramme of mercury, sp. gr. 13·6; find the sectional area of the tube.

20. The specific gravity of a faulty iron casting which weighs 3 lbs. is found to be 5·8. If the normal specific gravity of cast iron be 7·2; find what volume of the faulty iron is unoccupied by iron.

(A cubic inch of water weighs 0·57 oz.)

21. If the specific gravity of a mixture of glycerine and water be 1·034, find the relative weights of glycerine and water in the mixture, the specific gravity of pure glycerine being 1·26.

22. The mass of a piece of brass is 25 grammes and the density of brass is 8·4 grammes per c.cm., find the volume of the brass.

CHAPTER II.

FLUID PRESSURE.

12. General considerations on Stress.

Consider a block of wood lying on a smooth horizontal table, let a weight be placed on the wood, as in Fig. 1. The wood is in equilibrium, the downward force exerted by the weight is balanced by the upward force between the table and the wood. Imagine the block divided into two parts by a horizontal plane CD, the upper part is acted on by the weight which presses it downwards; since there is equilibrium the weight must be balanced by a force exerted upwards by the lower part of

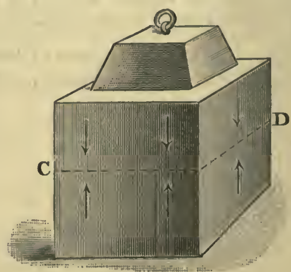


Fig. 1.

the block on the upper; if now we consider the lower part of the block a downward force is exerted on its upper surface equal to the upward force which this surface exerts on the upper portion. This downward force is balanced by the upward force exerted by the table; the two portions of the block are squeezed together across the section CD; the block is said to be under **Stress**. The two equal forces acting in opposite directions on the two portions of the block across the

section CD constitute a **Stress**. In the case considered the forces are at right angles to the surfaces on which they act; the external forces, the weight and the pressure of the table, are such as to bring the two portions of the body, separated by the plane CD, more close together than they could otherwise be; each portion of the body "thrusts" or pushes the other. We speak of the force which each part of the body exerts on the other across the plane CD as a normal Thrust or more simply as a "**Thrust.**"

13. Thrust and Tension.

The Stress which we have just been considering consists of a simple Thrust acting in opposite directions on the two sides of any horizontal plane such as CD, by which we imagine the body to be divided.

It should of course be noticed that the division is imaginary, the two parts of the body on either side of any horizontal plane thus act on each other; it is not necessary actually to cut or divide the body to give rise to the action.

There are, of course, many other ways in which we can apply a simple thrust to a surface; thus when we push a body with a long pole, directing the push along the axis of the pole, any section of the pole at right angles to the axis is subject to a Thrust, the portion of the pole on one side of the section thrusts and is thrust by that on the other, the stress across the section is a simple thrust.

But now suppose one end of the pole is attached to some body and that we pull at the other; if we now consider a section of the pole at right angles to its length, the portion of the pole on one side of the section is pulled by that on the other; the pole throughout its length is subject to a **Pull or Tension** instead of a Thrust.

The Stress across each section at right angles to the length is a simple **Tension** acting along the pole.

In both these cases however the stress is at right angles to the surface to which it is applied.

14. Shearing Stress.

Let us return now to the block of wood on the table and imagine it divided, as in Fig. 2, by a plane *CD* inclined to the horizon. The upper part is acted on by the weight which presses it vertically down; since there is equilibrium this vertical force must be balanced by the force which the lower part of the block exerts on the upper, this latter force then must be vertical. But the plane *CD* is inclined to the horizon, hence in this case the force which the lower portion of the block exerts on the upper is inclined to the plane across which it acts. The force therefore may be resolved into two components, one, at right angles to the plane, constituting a normal thrust on the upper part, the other, parallel to the plane, preventing the upper part from slipping down; this second component constitutes a "Shearing Force." It is balanced by the equal and opposite shearing force exerted on the portion of the block below the plane *CD* and these two forces constitute a Shearing Stress. The components of the stress across the plane *CD* are a normal thrust which tends to compress the solid into a smaller space and a shearing stress tending to make it slide parallel to the plane *CD* and thus to change its shape.

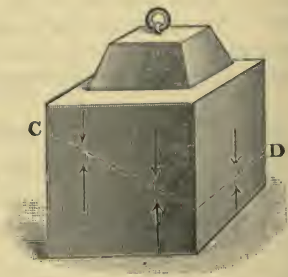


Fig. 2.

15. Stress in Solids and Fluids.

Hence, if we imagine any plane drawn in a solid body, the Stress across the plane due to the action which the portion of the solid on one side of the plane exerts on that on the other may be either a Thrust, a Tension or a Shearing Stress.

In consequence of its Rigidity a solid can withstand shearing stress, it yields slightly until the impressed force is

just balanced by the elastic forces called into play by the yielding and then retains its new form so long as the force is impressed.

A fluid however cannot permanently withstand shearing stress.

Consider a solid body *ABC*, Fig. 3, resting on a table and let *DE* be a plane inclined to the horizon, dividing the body into two parts. Then the weight of the portion *DBE* is balanced by the force across this plane. If this force were a normal thrust perpendicular to *DE* the portion *DBE* would slide down, its weight has a component parallel to the plane; this component however is balanced by the shearing force exerted across the plane.

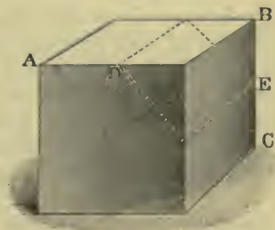


Fig. 3.

Now, suppose the portion *DBE* to become fluid; it runs away, the fluid has no rigidity and in consequence can exert no shearing force across any plane, there is therefore no force to balance the component of the weight parallel to *DE*, and equilibrium can no longer be maintained; the fluid yields to the impressed shearing force.

We have thus arrived at the following results. If we imagine a body to be divided into two parts, each part exerts a force on the other across the dividing surface.

These two forces are equal and opposite and are spoken of together as a **Stress.**

In general the forces can be resolved into components respectively at right angles to, and parallel to the surface across which they act.

And we can state the following Definitions.

DEFINITION. *The components of a* **Stress** *at right angles to the surface to which it is applied constitute a* **Thrust** *or a*

Tension. *The components parallel to the surface constitute a* **Shearing Stress.**

A **Solid** *is a body which offers permanent resistance to any form of stress, so long at least as the stress is not too great.*

If the stress be too great the solid may yield or break.

A **Fluid** *is a body which offers no permanent resistance to continued shearing stress, however small.*

It follows from this that any shearing stress, however small, will *in time* produce motion in a fluid—if the fluid be very viscous it will be a long time before flow takes place under a small shearing stress; if, on the other hand, the viscosity be small, flow follows rapidly. If there were no viscosity no shearing stress could ever be exerted whether the fluid were at rest or in motion.

DEFINITION. *A* **Perfect Fluid** *is a body which, whether at rest or in motion, can never offer resistance to a shearing stress, however small.*

There are no fluids known which satisfy this definition.

When however a fluid is in equilibrium, whether it be viscous or not, there can be no shearing stress, for since a fluid can offer no *permanent* resistance to shearing stress even a small shear, if it existed, would in time disturb the equilibrium. We thus arrive at the result, that *In any fluid in equilibrium there is no shearing stress.*

16. Fundamental Property of a Fluid.

The following Proposition then expresses the Fundamental Property of a fluid.

PROPOSITION 5. *To prove that, when a fluid is in equilibrium, the force, which it exerts on any surface with which it is in contact, is at right angles to that surface.*

For, let ACB, Fig. 4, be a portion of the surface and, if it

be possible, let the force P which the fluid exerts on a small portion of the surface near C be inclined to it in the direction DC. Then the force which the surface exerts on the fluid is P, acting in direction CD. This force can be resolved into a force R at right angles to the surface, constituting a normal thrust on the fluid, and a tangential force T parallel to the surface. This constitutes a shearing force and tends to make the fluid particles slide over the surface; since the fluid has no rigidity it cannot resist this force and motion will take place, which is contrary to the supposition that the fluid is at rest. Hence there can be no tangential force such as T, therefore the whole force is R, at right angles to the surface.

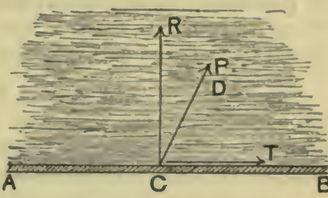


Fig. 4.

Thus the force exerted by a fluid on any surface with which it may be in contact, or by one portion of a fluid on any other portion, across any surface separating the two, is a normal thrust at right angles to the surface.

Experiments have shewn that it is possible for a fluid under certain circumstances to sustain a tension or pull; these circumstances however occur very rarely, for our present purposes we may suppose that the only stress which can exist in a fluid is a thrust.

17. Stress distributed over a Surface.

Imagine now that we have a piece of a stiff board resting on a table; on placing weights on the upper side of the board, force is exerted on the table, and this is balanced by the force[1] which the table exerts on the weights. Suppose the board is divided into a number of squares each 1 centimetre in edge and therefore 1 square centimetre in area; there is in general a force between each of these squares and the table, though the forces acting on each of the squares need not be equal. Suppose, however, that the same weight, 50 grammes say, is placed

[1] Part of this force is due to the weight of the board, we suppose this to be small compared with the weights it carries and neglect its effect.

on each square; then the upward force on each square will be 50 grammes weight; the resultant upward force will be found by multiplying by 50 the number of square centimetres contained in the area of the board.

The force in this case is **Uniformly Distributed** over the surface, and the thrust or resultant upward force is found by multiplying the force per unit area of the surface by the number of square centimetres in the area.

DEFINITION. *The* **Thrust** *on a surface is said to be* **Uniformly Distributed** *over the surface when it is the same on every equal area of the surface.*

In the example given above the thrust on each square centimetre is 50 grammes weight, it is uniformly distributed; but suppose that, instead of placing 50 grammes on each square centimetre, the weights were irregularly placed, so that on some squares there were more than 50 grammes, on others less, while the total weight carried remained the same; the total thrust would remain the same, but its distribution would be variable.

In the illustration it is of course possible that the thrust of 50 grammes weight which acts across each square centimetre may not be uniformly distributed over that square centimetre; if we suppose the square centimetre to be divided into a large number of very small equal areas and if the thrust on each of these areas be the same, then the distribution over the square centimetre is uniform, this case is included in the definition by the introduction of the word "every."

18. Pressure at a Point. It is found convenient to give a name to the thrust per unit area of any surface.

DEFINITION. *When a thrust is uniformly distributed over a surface, the thrust on each unit of area is called the* **Pressure at each Point** *of the surface.*

Now let P be a thrust which is uniformly distributed over a surface, let p be the pressure at each point of the surface, and let a square centimetres be the area of the surface.

Since on each square centimetre there is a thrust p, on a square centimetres the thrust is pa, but the total thrust is P. Hence

$$P = pa,$$

and
$$p = \frac{P}{a}.$$

Thus *The* **Pressure at each Point** *of a surface exposed to a uniform thrust is found by dividing the thrust on the whole surface by the number of units of area it contains.*

Now, even when the thrust is variable over the surface, we may treat it as uniform over a small area—a square centimetres—if that area be sufficiently small; and in this case, if P be the total thrust on the area, the ratio P/a gives the pressure at each point of the area.

DEFINITION. *When the thrust over any surface is not uniformly distributed the* **Pressure at each Point** *of the surface is the ratio of the thrust on a small portion of the surface which includes the point to the area of that portion when that area is sufficiently small.*

Thus, if P be the thrust on a surface of area a under variable pressure, p the pressure at each point of that surface, then

$$p = \frac{P}{a},$$

when a is taken so small that the thrust over the portion of surface considered may be treated as uniform.

19. Average Pressure. The ratio of the total thrust on any plane surface to the area of that surface is known as the **Average Pressure** at each point of the surface: if the total thrust be P, and the area of the surface a, then the average pressure is P/a.

If the thrust be uniformly distributed the average pressure and the pressure at each point are the same; if the thrust be variable, the pressure at any point is the average pressure on a portion of the surface containing the point, and so small that the distribution of thrust over that portion may be treated as uniform.

If a, the area of the surface, be unity, we see that the average pressure is equal to the thrust on the surface. Thus *The* Average Pressure *is the thrust per unit of area.*

20. Examples of Uniform and Variable Thrust.

If we consider a horizontal surface above which a mass of sand is piled, there will be a thrust on the surface, arising from the weight of the sand: if the sand be piled to a uniform depth all over, the thrust will be uniform, and the pressure at each point of the surface the same; if, on the other hand, the upper surface of the sand be uneven, the thrust will usually be variable and the pressure will differ from point to point.

Or again, consider a rectangular vessel filled with water having vertical sides and a horizontal bottom; the vertical forces acting are the weight of the water and the upward thrust of the bottom, these two are equal; moreover the thrust is uniformly distributed and the pressure is the same at each point of the base. The same would be true if the vessel contained a solid which just filled it, but the solid would exert no force on the sides of the vessel; when, however, it contains fluid each side is subject to a horizontal thrust the amount of which can be calculated. This thrust, it can be shewn, is not uniformly distributed, the force on any small portion of the surface near the top of the liquid is less than that on an equal portion near the bottom, the pressure is variable from point to point. Thus the pressure at the bottom of a dock of uniform depth is uniform, that on the dock gates is variable.

21. Units of Pressure.

The pressure at a point is found we have seen by dividing the thrust or normal force impressed on a definite surface by the area of that surface; we therefore speak of a pressure of so many units of force per unit of area; the numerical measure of a pressure depends on the unit of force and on the unit of area.

We may express it in dynes or in grammes-weight per square centimetre, or in poundals per square foot. A common unit of pressure adopted in England is pounds-weight per square inch.

Suppose, for example, it is found that the force acting on a surface 100 square centimetres in area is 50 kilogrammes weight, and that the pressure is uniform; the pressure at each point of the surface is 50/100 or ·5 kilogrammes weight per square centimetre. Similarly, if the force on a square 1/100 of a square inch in area be 2 lbs. weight, then the pressure is $2/\frac{1}{100}$ or 200 lbs. weight per square inch.

It must be clearly remembered that pressure is not force. In order to determine the thrust or force, impressed normally on a given plane surface by fluid pressure uniformly distributed, we must *multiply the pressure at each point by the area of the surface.* If the pressure is not uniform, the problem of finding the total thrust is more complex; if the average pressure at each point be known, the thrust is found by multiplying the average pressure by the area.

Examples. (**1**) *A surface is subject to a pressure of* 15 *lbs. weight to the square inch; determine it in grammes weight per square centimetre, and also in dynes per square centimetre.*

1 square inch contains $(2·54)^2$ or 6·45 sq. cm., 1 lb. contains 453·6 grammes.

Thus the thrust on an area of 6·45 sq. cm. is $15 \times 453·6$ grammes weight.

Hence the pressure is $15 \times 453·6/6·45$ grammes weight per square centimetre or 1055 grammes' weight per square centimetre.

Now the weight of 1 gramme contains 981 dynes.

Hence the required pressure is

$$1055 \times 981 \text{ or } 1·033 \times 10^6 \text{ dynes per sq. cm.}$$

Thus the pressure in question, which we shall see is about that exerted by the atmosphere, is equivalent approximately to a weight of 1 kilogramme per square centimetre; we might produce it by erecting a vertical tube 10 metres (1000 cm.) high and one square centimetre in area; if such a tube were filled with water the thrust on the base 1 sq. cm. in area would be the weight of 1000 c.cm. of water or 1 kilogramme.

(**2**) *The total thrust on a surface* 5 *square feet in area is found to be the weight of* 1 *ton, find the average pressure in lbs. weight per square inch.*

The surface contains 5×144 or 720 square inches, one ton is 2240 lbs.

Thus the pressure is 2240/720 or $3\frac{1}{9}$ lbs. weight per square inch.

(3) *Taking the atmospheric pressure at 15 lbs. weight per square inch, find the weight supported by a square mile of the earth's surface.*

One square mile is 4,014,489,600 square inches; the thrust in pounds weight is found by multiplying this by 15. It is therefore 60,217,334,000 lbs. weight.

22. Graphical Solutions.

The following graphical method of representing the pressure at any point of a plane surface is sometimes convenient.

We have seen that pressure is measured by the force impressed per unit of area: taking a square centimetre as the unit of area, the pressure may be given as so many grammes weight per square centimetre: now imagine the surface to be horizontal and erect on each square centimetre a vertical column of some homogeneous substance, of such a height that its weight may be equal to the force exerted by the fluid on the square centimetre which supports the column. Since the weight of a cubic centimetre of water is 1 gramme weight if the column be of water, h cm. in height, its weight will be h grammes; this is supported by the portion of the surface, 1 square centimetre, on which it rests, and if the weight of this column is to represent a pressure of p grammes weight per square centimetre, we must take h equal to p. Hence the pressure may be represented by the height of a column of water 1 sq. centimetre in area. The number of centimetres in the height of this column will be equal to the number of units of pressure in the pressure it represents.

The height of this column of liquid is sometimes spoken of as the "head" of liquid which gives the pressure. Thus we might speak of the pressure of the atmosphere, which is about 1 kilogramme weight per sq. cm., as due to a "head" of water 10 metres in height.

There is no need to select water as the substance by the aid of which the pressure is measured; it is, however, in most cases convenient to do so.

23. Pressure within a Fluid.

So far we have dealt with the thrusts which a fluid exerts on a surface with which it is in contact; we may compare these

to the force between a solid block such as that shewn in Fig. 1 above, and the table on which it rests. But we have seen (Section 15) that we must, in the case of the block, suppose stresses to exist throughout its substance. For a horizontal plane such as CD, Fig. 1, we have a normal thrust exerted in opposite directions on the two sides of the plane; if the plane be oblique as in Fig. 2, the normal thrust is accompanied by two tangential forces constituting a shearing stress.

In the same way stresses exist throughout the substance of any fluid.

Consider a mass of fluid in a vessel and suppose, for simplicity, that the sides of the vessel are vertical; imagine a horizontal plane CD, Fig. 5, drawn in the fluid. The forces acting on the fluid above this plane are its weight and the downward thrust of the atmosphere on its upper surface—these act in a vertical direction—together with the thrusts of the sides; the directions of these last forces are horizontal and they are in equilibrium among themselves. In order then that equilibrium may be maintained there must be an upward vertical thrust across the horizontal plane CD, equal to the sum of the weight of the fluid above the plane, and the downward thrust due to the atmosphere.

Fig. 5.

Or again, if the plane CD be inclined to the horizon, as in Fig. 6, there will, as in the solid, be a stress across it; this stress however will differ from that in the solid in that the forces which compose it are at right angles to CD. In the fluid there can be no stress along CD such as exists in the solid. In the fluid there is therefore a thrust across CD, this thrust has a vertical component which balances the sum of weight of the fluid above and the downward

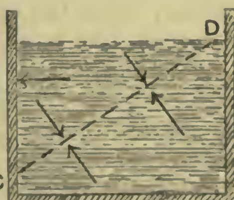


Fig. 6.

vertical thrust of the atmosphere; it also has a horizontal component and this balances the resultant horizontal thrust on the vertical sides of the vessel. In the solid the resultant force across CD, Fig. 2, is vertical; there is no force on the vertical sides and hence no horizontal component to the force across CD. There is, in consequence, a shearing stress in the solid across CD. The fluid cannot support such a stress, the wedge of fluid above CD, Fig. 6, would change in form and slide down were it not for the thrusts impressed on it by the sides of the vessel.

24. Pressure at a Point within a Fluid.

Consider now a small plane surface of area α immersed in a fluid; the fluid on either side of the surface exerts a thrust on the surface; if the fluid on one side could be removed it would be necessary to exert a force on that side in order to balance the fluid thrust on the other. Let the magnitude of this force be P. Then P measures the thrust in the fluid across the surface of area α.

The ratio P/α is defined as the **Average Pressure** at each point of the surface.

If the thrust over the surface be uniformly distributed then the ratio P/α is the **Pressure at each Point** of the surface. If the thrust over the surface be not uniformly distributed then, in order to find the pressure at any point, it is necessary to reduce the area of the surface until it is so small that the distribution of thrust over it may be treated as uniform; when this is the case the ratio P/α measures the pressure at any point of the surface.

DEFINITION. *In order to find the pressure at a point of a fluid, imagine a small plane surface of area α immersed in the fluid so as to contain the point. The* **Pressure at the Point** *is measured by the ratio of the thrust on one side of the surface to the area of the surface, when that area is so small that the distribution of thrust over it may be treated as uniform.*

The meaning of the term pressure at a point may perhaps be made clearer from the following. Let A, Fig. 7, be a point in a fluid at which the pressure is required. Imagine a small plane surface placed at A and a tube inserted in the fluid in such a way that the surface may form a piston in the tube; suppose further that it is possible for the piston to move without friction in the tube. Now let all the fluid be removed from the tube on one side of the piston: the thrust on the other side will drive the piston down the tube unless force be applied to it; suppose that a force P applied at right angles to the piston holds it in its place, then P measures the thrust on the piston and if a be the area of its surface P/a is the average pressure at each point of its surface. If the area of the piston be so small that the thrust may be taken as uniformly distributed P/a will be the pressure at each point.

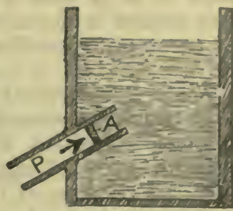


Fig. 7.

The arrangement described above does not constitute a practical means of measuring the pressure; it is not an experiment to illustrate the measurement of pressure; it is merely an illustration, impossible to realize in practice, of what is meant by fluid pressure. Practical means of measuring the pressure at a point will be given in Sections 36—40.

25. Pressure in different directions.

At any point in a solid imagine a small surface drawn, containing the point, and consider the thrust across this surface; it will of course depend on the forces which act on the solid; it will also, in general, depend on the direction within the solid in which the surface is drawn, thus, if the weight of the substance be the only force, there will be a normal thrust on a horizontal surface but no normal thrust on a vertical surface; the thrust depends on the direction of the surface. This is not the case in a fluid; the thrust on a surface of very small area placed at a given point is the same in whatever direction the surface be placed.

If, when the surface be horizontal, there be a thrust of P grammes weight upon it, then it can be shewn that there is the same thrust upon it when it is vertical, or in any other position, so long as it is sufficiently small and passes through the given point. In fact it follows from the fundamental definitions that,

The pressure at a point in a fluid is the same in all directions about the point.

The direction therefore of the small surface placed in the fluid so as to form a piston in the illustration given in the last section is immaterial, the thrust upon it is not changed by turning it about any point in itself.

This fundamental property can be deduced from the fact that there is no shearing stress in a fluid. The proof is given below (Proposition 6). A direct experimental proof would require somewhat complicated apparatus, the law however is involved in many of the experiments which will be described, and the student, who finds a difficulty in following the mathematical proof, may believe it because the results of experiment bear out theoretical deductions from the law.

The proof that the pressure is the same in all directions about a point is based on the following considerations. Suppose that all the particles of a fluid are acted on by some force, such as their weight, and consider the matter which lies within some surface drawn in the fluid; the resultant impressed force acting on this matter will be proportional to the volume of fluid within the surface, and this force is balanced by the thrusts on the surface arising from the fluid pressure.

Thus if, to make ideas definite, we consider a small cube in the fluid and suppose the weight of the fluid to be the only impressed force, the forces acting on the cube are the weight of the fluid which it contains and the six thrusts, one on each of the six faces of the cube; the resultant of these thrusts therefore must balance the weight. Now the thrust on each face is proportional to the area of the face, while the weight is proportional to the volume of the cube. Thus we have the resultant of six forces, which depend on the area of the faces, balancing a force which depends on the volume of the cube.

Suppose now that the cube is reduced in size so that each edge becomes—say—$\frac{1}{10}$th of its previous length, the faces will then become $\frac{1}{100}$th of what they were while the volume of the tube will be $\frac{1}{1000}$th of its previous value; the forces then which depend on the surface will be reduced 100-fold, those which are proportional to the volume will be reduced 1000-fold, or ten times as much; if the edge be again reduced to a

tenth, the forces proportional to the volume will again be reduced ten times as much as those which depend on the area of the surface.

Thus, proceeding in this manner, we see that we can make the forces which depend on the volume as small as we please when compared with those which depend on the surface; in the end, then, when the cube has become very small its weight may be neglected in comparison with the forces which arise from fluid pressure, and these forces form a system in equilibrium among themselves.

This statement is true whatever be the shape of the small portion of the fluid which we consider; let us apply it to a small triangular prism in the fluid.

*PROPOSITION 6. *To prove that the pressure at a point in a fluid is the same in all directions about the point.*

Let ABC, Fig. 8, be a small triangle in the fluid, draw lines AA', BB', CC' each l centimetres in length at right-angles to ABC; join $A'B'$, $B'C'$ and $C'A'$ and consider the portion of fluid within the prism thus formed.

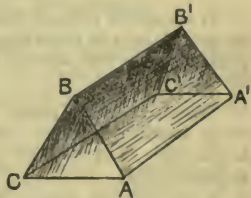


Fig. 8.

Let a, b, c be the lengths of the sides of the triangle BCA, p_1, p_2 and p_3 the average fluid pressures on the faces $BCC'B'$, $CAA'C'$, $ABB'A'$ respectively. The areas of these faces are respectively la, lb and lc square centimetres, hence the normal thrusts are lap_1, lbp_2, and lcp_3.

Now we have seen that when the prism is made very small, these three forces form a system in equilibrium among themselves; but, when three forces are in equilibrium, they can be represented by the sides of a triangle to which they are parallel.

The three forces in question are at right angles to the sides of the triangle ABC, hence, if this triangle were turned through a right angle in its own plane, its sides would be parallel to the directions of the forces; thus the three forces

are respectively proportional to the sides a, b, c of the triangle BCA.

The ratio then of each force to the corresponding side must be the same for all the forces. Now these ratios are

$$p_1la/a,\ p_2lb/b \text{ and } p_3lc/c.$$

Hence $$p_1l=p_2l=p_3l,$$

or $$p_1=p_2=p_3.$$

Thus the average pressure on each face is, when the prism is made very small, the same; but, in this case, the average pressure on a face is the pressure at the point, to which the triangle is reduced, estimated in the direction of the normal to that face: thus the pressures at right angles to the faces of any very small prism enclosing the point are ultimately equal, or in other words,

The pressure at a point in a fluid is the same in all directions about that point.

It should be noticed that the proof depends on the fact that the force on each face is at right angles to that face; if there were a shearing force parallel to the face as well as the simple thrust the proposition would not be true.

We may put the proof in mathematical form thus. Let $ABCC'B'A'$ be a small triangular prism in the fluid. Let the face $ACC'A'$ be horizontal. Let l be the length of the prism; a, b, c, the sides of the triangle BCA.

Let p_1, p_2, p_3 be the average pressure on the faces, and ω the weight of a unit of volume of the fluid.

Let d be the perpendicular distance of the vertex B from the base CA.

The volume of the prism is $\frac{1}{2}bdl$ and its weight is $\frac{1}{2}\omega bdl$; this force acts vertically and is therefore at right angles to the face $ACC'A'$.

The other forces are p_1al, p_2bl and p_3cl at right angles to the faces.

Resolve these vertically

$$p_2bl=\tfrac{1}{2}\omega bdl+p_1al\cos C+p_3cl\cos A.$$

Resolve the forces horizontally

$$p_1al\sin C=p_3cl\sin A.$$

But we know that $a\sin C=c\sin A$.

Thus $$p_1 = p_3,$$

also $$b = CA = a\cos C + c\cos A.$$

Hence substituting in the first equation

$$p_2 b = \tfrac{1}{2}\omega bd + p_1 (c\cos A + a\cos C)$$
$$= \tfrac{1}{2}\omega bd + p_1 b.$$

Thus $$p_2 - p_1 = \tfrac{1}{2}\omega d.$$

Now when the prism is made very small, so that p_1 and p_2 become the pressures in two different directions at a point, then d is indefinitely small. The difference therefore between p_1 and p_2 can be made as small as we please, or p_1 is equal ultimately to p_2.

Hence ultimately

$$p_1 = p_2 = p_3.$$

Now p_2 is the pressure in a vertical direction, p_1, p_3 pressures in any other two directions. Hence the pressure in a vertical direction is equal to that in any other, thus the pressure is the same in all directions about a point.

26. Transmissibility of Fluid Pressure.

If the pressure at any point of a fluid is changed, that at all other points is changed also; it follows, from the fundamental property of a fluid, that, for a liquid, the change of pressure at all points is the same.

A fluid in this respect differs from a solid. Imagine a cylinder fitted with a piston and place in it a portion of a solid which just fits the cylinder loosely. Put weights on the piston, the force thus applied to the top of the solid is transmitted to the base; unless the solid expands laterally under the force, so as to fit the cylinder more tightly, there will be no pressure on the sides; if, however, the substance in the cylinder is a fluid this is no longer the case; the addition of the weights increases the pressure or force per unit of area on the top of the fluid, this increase is transmitted by the fluid in all directions; the pressure at each point is increased and, as we shall shew, for a liquid, the increase of pressure is the same at all points.

This may be illustrated by the following arrangement.

Imagine a vessel fitted with a number of openings, each closed by a piston, as shewn in Fig. 9; suppose the whole to be filled with liquid. In order to keep the pistons in their place a force must be applied to each. Suppose now that the force on any one piston is increased, the other pistons will be driven out, and in order that equilibrium may be maintained it is necessary that additional force should be applied to each of them. *If we could secure frictionless pistons, all of the same area, it would be found that the additional force applied to each piston would be the same;* if, however, the pistons differ in area, the force necessary to maintain any piston in position would be found to be proportional to the area of the piston; the ratio of the force to the area over which it is applied is the same for all.

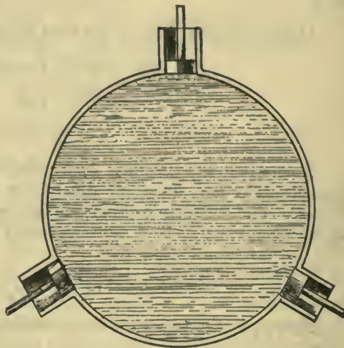


Fig. 9.

An increase of fluid pressure applied at one point is transmitted equally to all other points.

The experiment in this form is impossible; we cannot obtain frictionless pistons. The principle, however, is illustrated by the action of various pieces of apparatus which will be described shortly, see Sections 27, 28, and by some experiments which will be better understood when we have considered some of the methods for measuring fluid pressure. We proceed now to give a formal proof of the principle.

PROPOSITION 7. *An increase of pressure, at any point of a liquid at rest, is transmitted without change to every other point.*

For let A, B, Fig. 10, be two points within the liquid.

(i) Suppose first that the line AB lies entirely within the liquid.

Construct a small cylinder, having the line AB for its axis, and consider the forces acting on the fluid within this cylinder. They are

(1) the thrusts on the ends A and B, parallel to the axis AB,

(2) the thrusts on the curved surface at right angles to the axis,

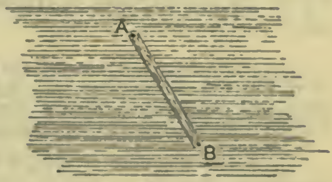


Fig. 10.

(3) the resultant of the external impressed force.

Now the liquid is in equilibrium and the thrusts on the curved surface have no component parallel to the axis.

Thus the difference between the thrusts on the two ends must balance the component of the impressed force in the direction of the axis.

But this component remains the same even though the pressure be changed.

Hence the difference between the thrusts on the ends A and B is a constant; but the areas of these ends are equal.

Thus the difference between the pressures at the two ends is always the same.

Hence, if by any means the pressure at the point A is increased, that at B is increased by the same amount, otherwise the difference between the two would change and it has just been proved that this difference is unchanged.

(ii) Suppose that the line AB does not lie entirely within the liquid.

Join the points A and B by a series of straight lines AP, PQ, QB, Fig. 11, etc. each of which does lie entirely in the liquid. Then the proposition just proved holds for each of the pairs of points A, P . P, Q etc.

Hence if the pressure at A be increased, that at P is increased equally; but if the pressure at P is increased that at Q is increased equally, and so for all the points. Hence the increase of pressure at B is equal to that at A.

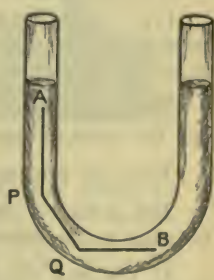


Fig. 11.

Thus any increase of pressure produced at any point of a liquid in equilibrium is transmitted without change to every other point.

The proposition is not true, for a gas or compressible fluid, in any case in which it is necessary to take into account the weight or other force impressed on the gas; for if a gas be subject to increased pressure its volume is diminished and its density is increased.

The weight therefore of the gas within the cylinder AB is changed by the change of pressure, and in consequence the difference of pressures between the two ends is changed also.

The density of a gas is however usually very small, the weight therefore of a limited portion is generally small compared with the force to which each unit of area of its surface is subject; for many purposes we may omit the consideration of the weight of the gas entirely and may suppose that in the case of a gas we are dealing with a fluid acted on throughout its mass by no impressed forces. We may shew that in this case the pressure is the same at every point.

PROPOSITION 8. *To prove that if a fluid be acted on by no impressed force the pressure is the same at every point.*

Let A, B, Fig. 12, be two points in such a fluid. Join AB and suppose the line AB to be entirely within the fluid. Construct a small cylinder about AB as axis. The cylinder is in equilibrium under

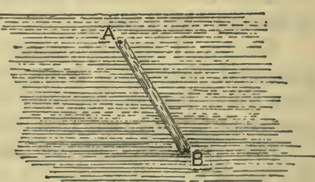


Fig. 12.

(1) The thrusts on the two ends acting parallel to the axis.

(2) The thrusts on the curved surface acting at right angles to the axis.

Hence the thrusts on the two ends are equal and opposite; but the areas of the ends are the same.

Thus the pressures at the two ends are equal.

But A and B are any two points in the fluid. Hence the pressure is the same at any point in the fluid.

If the line AB does not lie entirely in the fluid the proof can be extended as in Proposition 7 (ii).

Thus, provided the volume of gas considered is so small that its weight may be neglected, we may take the pressure in a gas to be the same at all points.

The above statement would not of course apply to the pressure throughout any large volume of a gas such as the atmosphere. The pressures at the top and bottom of a mountain are very different. Delicate pressure gauges will enable us to detect the difference in pressure between the attics and the basement of a house.

27. Hydrostatic Bellows.

This apparatus, designed by Pascal, illustrates the principle of the transmissibility of pressure in a fluid. A stout bladder, such as is used for a football, or a leather bellows, is attached to a piece of tube. The tube is fixed in a vertical position and the bladder rests on the table. A piece of light board is placed on the bladder and a weight rests on the board.

Water is then poured down the tube; the water flows into the bladder, causing it to expand and raise the weight; the level of the water in the tube stands, as at *C*, Fig. 13, some distance above the level of the board.

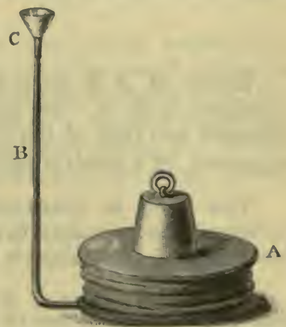


Fig. 13.

To explain the action, we notice that the weight is supported by the upward thrust of the water on the underside of the board; this upward thrust depends, partly on the pressure of the water and partly on the area of the surface of the board which is in contact with the bladder, and its value is obtained by finding the product of the two; if, therefore, the area in contact with the board be large, the upward thrust may be considerable, even though the pressure is not large. Suppose now, when the whole is in equilibrium, the level of the water in the tube is at *B*. Let more water be poured into the tube, and suppose that the weight does not rise.

The downward thrust over the surface of the water at B is increased by the weight of the water poured in, the pressure therefore at B is increased.

Hence, if w be the weight of water poured in and a the area of the section of the tube, the increase in the thrust on the area a at B is w; thus the increase in pressure is w/a. This increase of pressure is transmitted equally to all points, hence the upward pressure at all points of the under surface of the board is increased by w/a. Let the area of this surface be A, then the total upward thrust is increased by Aw/a.

In order that the board may not rise the weight upon it must be increased by an amount W equal to Aw/a. If the weight be not increased the board will rise and the level of the water in the tube will sink, thus reducing the pressure until equilibrium is again established.

In this arrangement we see that a weight w of water can support a weight W placed on the board and that

$$W = \frac{wA}{a}.$$

Hence, if A is large compared with a, W will be large compared with w. By making the area of the board considerable and that of the tube small, a large weight W can be supported by a small weight w of water.

This fact is sometimes described as the hydrostatic paradox: the principle involved in the above experiment is made use of in Bramah's Press (see Section 94).

Example. *The area of the tube used in an experiment like that described in Section* 28 *was* 10 *square millimetres, the area of the board* 100 *square centimetres. If* 10 *grammes of water are poured into the tube, find the additional weight which the board can support.*

We have 10 sq. mm. = ·1 sq. cm.

Thus the increase of pressure is 10/·1 or 100 grammes weight per square centimetre. The increased upward thrust on the board 100 sq. cm. in area is 100×100 or 10000 grammes weight. Thus if the board is not to rise an additional downward force of 10 kilos weight must be applied to it.

28. Illustrations of Fluid Pressure.

The apparatus shewn in Fig. 14 again illustrates the transmissibility of fluid pressure. In it M and N are two cylinders of different diameters which are filled with water and communicate through a tube at the bottom. They are fitted with pistons; the pressure at any point of the two pistons is the same; the upward thrusts on the pistons are proportional to their areas; thus, if a downward force w be applied to the smaller piston, a larger force W must act on the larger piston in order to maintain equilibrium.

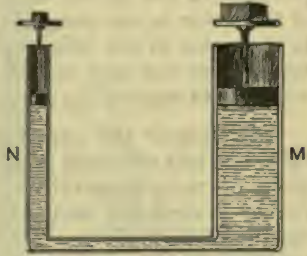


Fig. 14.

If the pistons be assumed to be frictionless, the relation of W to w is found thus:

Let A, a be the areas of the two pistons respectively. Then on the smaller piston there is a downward thrust w, the thrust per unit area of the piston is therefore w/a; this then is the pressure in the fluid and it is transmitted to each unit of area of the piston A. The total upward thrust then on this piston is Aw/a and this upward thrust must balance W. Hence

$$W = \frac{Aw}{a},$$

or

$$\frac{W}{A} = \frac{w}{a}.$$

In consequence of the friction, however, the relation of W to w given by experiment would differ from this.

Numbers of other illustrations of the effects of fluid pressure can be given. Thus

(i) Fill with water, a glass tube 30 or 40 cm. in length,

closed at one end, and place it with its open end downwards, in a vessel of water; the water remains in the tube, as shewn in Fig. 15; the pressure of the air on the free surface of the water in the vessel is transmitted to the water in the tube. The upward thrust over the open end of the tube is sufficient to support the contained water.

Fig. 15.

(ii) Repeat the experiment with a tube, one end of which is closed with a piece of thin india-rubber, the india-rubber is stretched and takes the form shewn in Fig. 15, the pressure on its upper surface is greater than that exerted by the water on the lower surface.

(iii) Again, take a hollow cylindrical vessel, such as a tin can or bucket, and place it bottom downwards in a vessel of water, the can floats; to sink it under water force is necessary, the upward thrust arising from the fluid pressure is greater than the downward force, the weight of the vessel, hence it floats; if a small hole be bored in the bottom the water spouts up in a jet.

(iv) Place a small beaker or tumbler mouth downwards in water, as in Fig. 15 *a*, and depress it below the surface; the water rises in the beaker, compressing the air it contains; the greater the depth to which the beaker is lowered, the greater will be the compression; the pressure in the water increases with the depth.

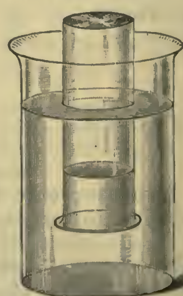


Fig. 15 *a*.

(v) The water supply of a town usually comes from a reservoir at some height above the town, in consequence the water in the pipes is under pressure; a jet of water allowed to flow from a hose-pipe will rise to a height which depends partly upon this pressure.

(vi) If a fluid be allowed to escape from a tall vessel through a hole in the side, the velocity with which it flows out depends on the pressure; if holes be made at various depths, as in Fig. 16, in the side of the vessel, the water flows more rapidly from the lower holes than from those above; the pressure is greater at the greater depth. This can be shewn by measuring the amount which flows from each hole in a given time; the level of the water in the vessel must be kept constant by some arrangement during the experiment.

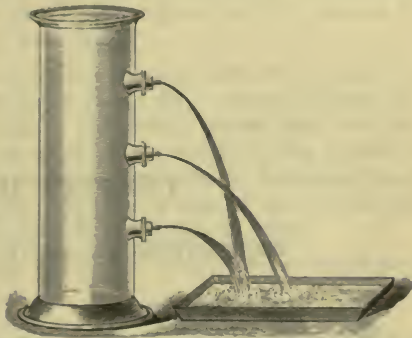


Fig. 16.

(vii) Grind the end of a glass tube flat, cover it with a plate of flat glass and hold the glass against the bottom of the tube with a string, as shewn in Fig. 17. Lower the whole some depth into a vessel of water and release the string, the glass remains in contact with the end of the tube and does not fall off. The upward thrust, due to the pressure of the water on the bottom of the glass, is more than sufficient to counterbalance its weight.

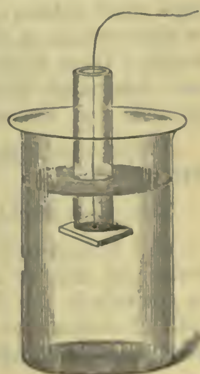


Fig. 17.

In the next chapter we give some fundamental propositions on fluid pressure when the only force acting on the fluid is its weight. We then describe some experiments on the measurement of fluid pressure and the numerical verification of some of the laws.

CHAPTER III.

PROPOSITIONS ON FLUID PRESSURE.

29. Fluid Pressure.

We assume in the following propositions[1] that the only forces, impressed on the portion of fluid which we consider, are the thrusts due, either to the action of surrounding fluid, or to solids with which the fluid is in contact, together with the weight of the portion of fluid considered.

If then we take any portion of the fluid, say that within some small sphere or cylinder, described in the fluid, the forces on this portion of fluid are the thrusts on its surface and its weight; these forces must form a system in equilibrium: we can determine from this the relation between the fluid pressure and the weight.

Proposition 9 deals with the pressure at points at the same level in a fluid.

Two cases of this proposition arise.

(i) It may be possible to join the two points by a straight horizontal line or a series of straight horizontal lines which lie entirely in the fluid; thus any two points in an ordinary bath of water could be joined by a straight line. Suppose, however, that there is a sponge or a piece of soap in the bath,

[1] Similar propositions may be proved in a very similar way for a fluid at rest under other forces than its weight. For these the reader is referred to Greaves' *Elementary Hydrostatics* (Cambridge University Press).

then a point on one side of the soap cannot be joined by one straight horizontal line to a point on the other side without cutting the soap; we can however join the two points by means of two or more such lines.

(ii) It may be impossible to join the two points by horizontal lines lying entirely in the fluid; thus, if there is a vertical partition stretching completely across the bath and reaching part way down, a point on one side of this cannot be connected with a point on the other side by horizontal lines lying entirely in the fluid; the same is true of two points, one in each leg respectively, in a fluid filling a U tube.

Proposition 9 applies to the first case; the second is dealt with in Section 30.

PROPOSITION 9. *If a fluid be at rest under the action of gravity, the pressures are equal, at any two points which can be joined by a single straight horizontal line lying wholly within the fluid, or by a series of such lines.*

In Fig. 18, let A, B be the two points in the same horizontal plane.

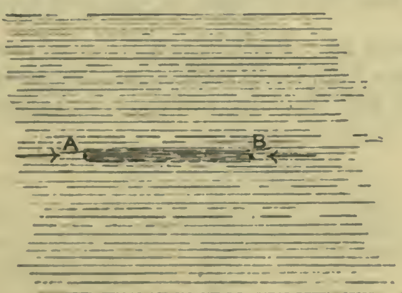


Fig. 18.

(i) Suppose that the straight line AB lies wholly within the fluid.

About AB construct a cylinder of very small section with its ends at right angles to AB and consider the forces impressed on the cylinder.

They are

(i) The thrusts on the ends A, B which act n opposite directions along AB.

(ii) The thrusts on the curved surface of the cylinder which are everywhere at right angles to AB.

(iii) The weight of the fluid within the cylinder, the line of action of which is vertical and therefore at right angles to AB.

Thus the only impressed forces in the direction of AB are the thrusts at A and B; these forces then must be equal and opposite; but the area of the end at A is equal to that at B.

Hence the fluid pressure at A is equal to that at B.

(ii) If AB cannot be joined by a single horizontal straight line lying wholly within the fluid, but by a series of such lines AP, PQ, ... etc., each of which does lie in the fluid, then the proposition is true for each of these lines.

Hence the pressure at A is equal to that at P, the pressure at P is equal to that at Q, and so on; thus the pressures at A and B are equal.

The next proposition deals with the pressures at two points in a fluid, one of which is vertically below the other.

PROPOSITION 10. *To find the difference of pressure between two points in a fluid, one of which is vertically below the other.*

Let A, B, Fig. 19, be the two points and suppose that the line AB lies wholly in the fluid.

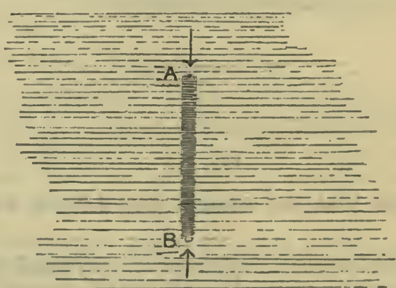


Fig. 19.

Consider a small vertical cylinder on AB as axis with its ends perpendicular to AB, let a be the area of either end and let p, p' be the pressures at A and B respectively.

The forces on the fluid composing this cylinder are

(i) The thrusts on the curved surface; these are horizontal, at right angles therefore to AB.

(ii) The thrusts on the ends; the values of these are pa and $p'a$ respectively; they act vertically in opposite directions parallel to AB.

(iii) The weight of the fluid contained in the cylinder; the direction of this force is also vertical, and parallel therefore to AB.

Thus the difference between the thrusts on the ends must balance the weight of the fluid in the cylinder.

Hence $p'a - pa =$ weight of fluid in the cylinder.

Suppose now that the fluid is homogeneous, so that its density is the same throughout; let ω be the weight of a unit of volume; let h, h' be the depths of the points A and B below some fixed horizontal surface.

Then $$AB = h' - h.$$

Now the volume of the cylinder is $AB \times a$ and its weight is $\omega \,.\, AB \,.\, a$ or $\omega (h' - h) a$.

Hence $$p'a - pa = \omega (h' - h) a.$$

Therefore $$p' - p = \omega (h' - h).$$

Now $\omega (h' - h)$ is the weight of a column of fluid of unit cross section and of height equal to the vertical distance between the two points.

Hence *The difference of pressure, between two points in the same vertical line, is equal to the weight of a column of fluid of unit cross section, and of height equal to the distance between the points.*

Thus in a homogeneous fluid the difference of pressure between two points in the same vertical line is proportional to the distance between the two points.

Corollary. If the fluid be not homogeneous it is still true that the difference of pressure is the weight of the column of fluid of unit cross section which extends from one point to the other.

30. Pressure at various points in a heavy fluid.

By combining the two propositions just proved we can shew (i) that *In any homogeneous fluid, the pressures at any two points in the same horizontal plane are equal*, and (ii) that *The difference of pressure between any two points is the weight of a column of fluid, of unit cross section, whose height is the vertical distance between the points.*

For suppose that A, B, Fig. 20, be two points in a fluid in the same horizontal plane which, however, because of some barrier cannot be joined by a horizontal line entirely within the fluid.

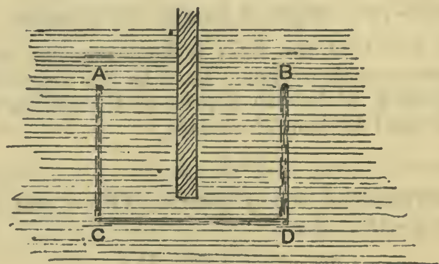


Fig. 20.

Draw AC and BD vertical and let C and D be two points in the same horizontal plane below the barrier which can be so joined.

Join CD. Then, since AB and CD are both horizontal, the distance AC is equal to BD.

The pressure at A is less than that at C by the weight of a column of unit cross section and of height AC.

The pressure at B is less than that at D by the weight of a column of unit cross section and of height BD.

The weights of these two columns are equal, and the

pressure at C is by Proposition 9 equal to that at D, for CD is horizontal.

Hence the pressure at A is equal to that at B.

If it be not possible to pass from A to B, by a single step of the nature indicated, it can always be done by a series of such steps.

To prove (ii) let A, B, Fig. 21, be any two points. Draw BC vertical and from A draw AC horizontal to meet BC in C.

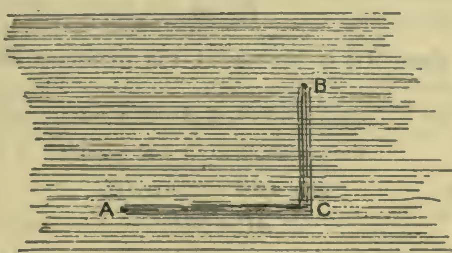


Fig. 21.

Then, by Proposition 10, the pressure at C exceeds that at B by the weight of a column of fluid of unit cross section and of height BC, while by Proposition 9, the pressure at A is equal to that at C.

Hence the pressure at A exceeds that at B by the height of a column of fluid of unit cross section and of height equal to the vertical distance between the points.

If it be not possible to pass from A to B by a single step of the nature just described, it can always be done by a series of such steps.

PROPOSITION 11. *The surface of a liquid, subject to constant pressure and at rest under gravity, is horizontal.*

Let A, B, Fig. 22, be two points in the same horizontal plane in a liquid at rest under gravity. Let π be the constant pressure to which the surface is subject and ω the

weight of unit volume of the liquid. Draw AC and BD vertically upwards to meet the surface in C and D.

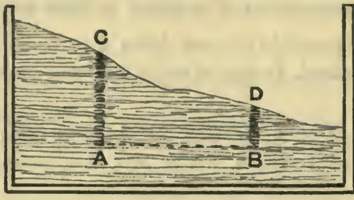


Fig. 22.

Then, since AB is horizontal, the pressures at A and B are equal.

But, the pressure at $A = \pi + \omega AC$,

and the pressure at $B = \pi + \omega BD$.

Hence $\pi + \omega AC = \pi + \omega BD$.

Thus $AC = BD$.

Therefore CD is always parallel to AB; hence the surface is horizontal.

31. Level Surfaces.

We shall shew later that the atmosphere exerts pressure on the earth's surface. (Section 67.)

The pressure of the atmosphere over a limited area of the earth's surface is, at any moment, a constant pressure; thus the free surface of water in a pond or reservoir is level.

Again, the various propositions just proved are true whatever be the form of the vessel in which the liquid is contained, so long at least as there is free communication between its parts. If water be contained in a bent tube with two vertical limbs, the level of the water in the two limbs of the tube is always the same. Moreover the pressures at any two points in the same horizontal plane, one in each limb it may be, are the same.

If π be the atmospheric pressure on the surface and ω the weight of a unit of volume of the liquid, then the pressure at

a depth h is $\pi + \omega h$. So long as π remains the same the pressure depends only on the depth of the point below the surface.

32. Effective Surface.

We shall find that in a number of cases we need not consider the atmospheric pressure at all, in many others it is necessary to do so. Now we can always represent a pressure as equal to the weight of a column of water—or of some other liquid—of unit cross section and of a definite height; a pressure of 1 kilogramme weight per square centimetre, for example, is equal to the weight of a column of water 1000 centimetres in height and 1 square centimetre in cross section.

Thus we may represent the atmospheric pressure as due to the weight of a column of water; if H_0 be the height of this column and ω the weight of a unit of volume of water, then

$$\pi = \omega H_0.$$

The pressure at a depth h below the surface of water then is

$$p = \pi + \omega h = \omega H_0 + \omega h = \omega (H_0 + h).$$

Suppose now that we consider a horizontal surface at a height H_0 *above* the surface of the water, then $h + H_0$ is the depth of the point below this surface; this imaginary surface is sometimes spoken of as the **Effective Surface**, and we see that the pressure at a point is proportional to the depth of the point below the effective surface. In this case the height H_0 is called the height of the water barometer (see Section 76).

We may, it is clear, without affecting the circumstances within the water, suppose that its surface is covered with a layer of water of sufficient depth to produce over that surface the pressure which actually exists there; the upper boundary of this layer being free from pressure, and imagine that the thrust over the actual surface is due to the weight of this superposed water and not to the atmosphere. If we are considering the pressure in some other liquid, not water, it will be most convenient to suppose the superincumbent layer to consist of this same liquid.

PROPOSITION 12. *Two liquids which do not mix are placed in a vessel, to find the pressure at a point in the lower liquid.*

Let A, Fig. 23, be a point in the lower liquid. Let

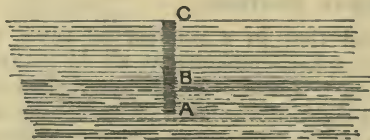


Fig. 23.

ABC drawn vertically from A meet the common surface of the two liquids in B and the upper surface of the upper liquid in C. Let $AB = h$, $BC = h'$. Let ω and ω' be the weights of unit volume of the lower and upper liquids respectively, p, p' the pressures at A and B, π the pressure at C.

Then from the upper liquid we have

$$p' = \pi + \omega' h',$$

and from the lower liquid

$$p = p' + \omega h.$$

Hence $$p = \pi + \omega h + \omega' h'.$$

The pressure at A is the pressure at the surface together with the weight of a column of unit cross section reaching from A to the surface.

* COROLLARY. *The common surface of two liquids which do not mix is horizontal.*

Let D, Fig. 23 a, be a point in the lower liquid at the

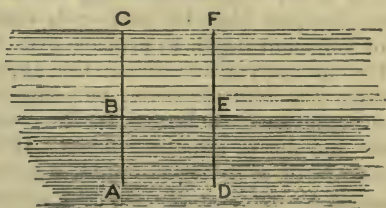


Fig. 23 *a*.

same level as A, and let DEF drawn vertically from D meet the common surface in E and the upper surface in F.

Then, since the free surface is horizontal, FC is parallel to DA.

Hence $$DF = AC.$$

Therefore $$DE + EF = AB + BC.$$

Hence multiplying both sides by ω'

$$\omega' . DE + \omega' . EF = \omega' . AB + \omega' . BC,$$

or $$\omega' (DE - AB) = \omega' (BC - EF) \quad\text{............ (1).}$$

Again, the pressure at D is equal to that at A.

Hence $\pi + \omega' . EF + \omega . DE = \pi + \omega' . BC + \omega . AB.$

Thus $$\omega . DE + \omega' . EF = \omega . AB + \omega' . BC,$$

or $$\omega (DE - AB) = \omega' (BC - EF) \quad\text{...............(2).}$$

Hence from (1) and (2)

$$\omega (DE - AB) = \omega' (DE - AB)$$

and this is impossible unless $DE - AB = 0$.

Therefore $DE = AB$, and BE is always parallel to AD.

But AD is horizontal, thus BE is horizontal.

The following experiment proves the truth of Proposition 11.

Experiment 5. *To prove that the surface of a liquid at rest under gravity is horizontal.*

Suspend a plumb-line above a vessel of water and observe the reflexion of the thread in the water, the reflected image is found to be in the same straight line as the thread. The thread therefore is at right angles to the reflecting surface[1]; but the thread is vertical, hence the surface must be horizontal.

33. A Liquid finds its own level.

The law that the free surface of a liquid at rest is horizontal, or, as it is sometimes put, that a liquid finds its own level, is illustrated in many ways.

[1] This result is made use of in Astronomy and Surveying to determine the *direction of the vertical at a given point*. See *Light*, Section 23.

Thus when as in Fig. 24 water is contained in a number of vessels of various shapes and sizes which all communicate together, the level of the water is the same in all.

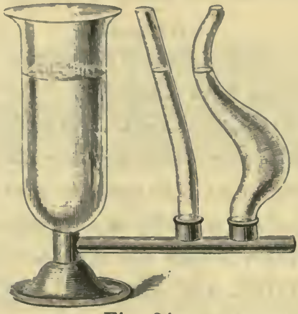


Fig. 24.

The Water Level shewn in Fig. 25 is another illustration of the principle. A long tube is bent at right angles at its

Fig. 25.

two ends; to these small glass vessels are attached, the tube is mounted at its centre on a universal joint and is filled with water slightly coloured. The water rises to the same level in each of the glass vessels; and an observer, placing his eye so as to look along the surfaces of the water, can read differences of level on distant scales towards which the apparatus is pointed in turn.

Fig. 26 shews the tube of a spirit-level; this consists of a

Fig. 26.

closed glass tube nearly filled with alcohol; the bubble of air left in always rises to the highest part of the tube; the tube is slightly bent and is mounted as in Figs. 27 (*a*) and (*b*) with its convex side uppermost, and in such a manner that, when the surface on which the instrument is placed is horizontal, the bubble of air may rest between two marks on the glass. The instrument can then be used so as to place in a horizontal position a surface, the level of which is adjustable.

EXPERIMENT 6. *To level a given plane surface with a spirit-level.*

The surface rests on three screws and by adjusting these its level can be altered.

First place the level so that its length may be parallel to the line joining two of the screws as in Fig. 27 (*a*). Adjust

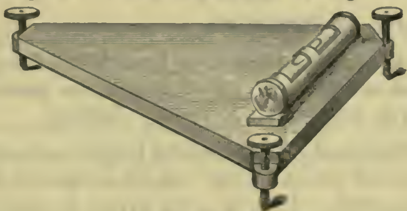


Fig. 27 *a*.

one of these screws until the bubble is in its central position; then place the level, as in Fig. 27 (*b*), at right angles to its

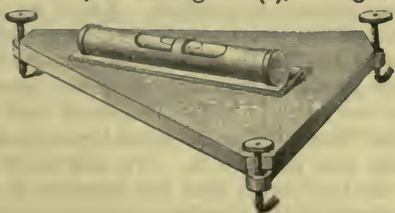


Fig. 27 *b*.

former position and adjust the surface by the third screw; when the level again reads correctly the surface is horizontal. Test this by placing the level in some other position.

A circular spirit-level is often employed. This is shewn in Fig. 28. A small cylindrical metal box is nearly filled with alcohol and closed by a glass top. The under surface of the glass is slightly concave, so that a bubble of air left above the alcohol rests in the centre when the bottom of the instrument is level. To use the instrument it is placed on the surface to be levelled, and this is adjusted until the bubble rests in its central position.

Fig. 28.

A surface which it is required to level and which is not fitted with levelling screws may be conveniently adjusted by the aid of three wedge-shaped slips of wood which are pushed under it or withdrawn from it as required.

34. Units of Pressure.

In the expressions which have been found for the pressure at a point, the units, in terms of which the pressure is to be measured, have not been specifically defined.

These will depend on the unit of length and on the unit of force in terms of which ω, the weight of unit volume of the liquid, is measured. Thus if ω be given in grammes weight per cubic centimetre, and h be measured in centimetres the pressure will be in grammes weight per square centimetre. If ω be in absolute C.G.S. units or dynes per cubic centimetre p is in dynes per square centimetre. Again we may use the English system of units and measure ω in pounds weight per cubic foot; then the depth h must be measured in feet and the pressure is given in pounds weight per square foot. Or we may adopt an entirely different system and measure the pressure by the "head" of some given liquid such as water or mercury which will produce it. We can find the head required from the knowledge that the pressure is equal to the weight of a column of the standard liquid of unit cross section and of height equal to the head.

Examples of both these methods will be given below.

It is sometimes convenient to express the pressure in terms

of the density of the liquid rather than of its weight per unit volume.

Now we know that, adopting the C.G.S. system, the weight of a body in dynes is found by multiplying its mass in grammes by g, the acceleration due to gravity. Thus we have the result that, if W be the weight in dynes of a body whose volume is V cubic centimetres and mass M grammes, and ρ be its density in grammes per cubic centimetre, then,

$$W = Mg = \rho g V.$$

If the volume of the body be unity then the weight W becomes ω, the weight of unit volume, and we see that

$$\omega = g\rho \text{ dynes per cubic centimetre.}$$

Thus the equation $p = \pi + \omega h$ becomes

$$p = \pi + g\rho h.$$

When this equation is used it is assumed that p and π are to be measured in dynes per square centimetre, h being in centimetres and ρ in grammes per cubic centimetre.

If we are employing the F.P.S. system a similar equation holds, ρ being in pounds per cubic foot, h in feet and p and π in poundals per square foot.

Examples. (1) *If the pressure of the atmosphere be taken as* 15 *lbs. weight per square inch and the weight of a cubic foot of water as* 62·5 *lbs. weight, find the pressure at depths of* (i) 10 *inches*, (ii) 20 *feet*, (iii) 50 *fathoms*, (iv) 1 *mile under the surface of water.*

(i) The pressure on each square inch of the surface is 15 lbs. weight. The weight of 1 cubic inch of water is 62·5/1728 or ·03617 pound weight.

Hence the weight of a column of water 1 square inch in area, 10 inches in height, is 10 × ·03617 or ·3617 lbs. weight.

Thus the pressure at a depth of 10 inches is 15 + ·3617 or 15·3617 lbs. weight per square inch.

(ii) 20 feet = 240 inches.

The weight of a column of water 1 square inch in cross section, 240 inches in height, is

240 × ·03617 or 8·68 lbs. weight.

Hence pressure at a depth of 20 feet is

15 + 8·68 or 23·68 lbs. weight per square inch.

(iii) 50 fathoms = 300 feet = 3600 inches.

The weight of a column of water 1 square inch in cross section, 3600 inches in height, is

$$3600 \times \cdot 03617 \text{ or } 130 \cdot 2 \text{ lbs. weight.}$$

Hence pressure at a depth of 50 fathoms is

$$15 + 130 \cdot 2 \text{ or } 145 \cdot 2 \text{ lb. per square inch.}$$

(iv) 1 mile = 63360 inches.

The weight of a column of water 1 square inch in cross section, 63360 inches high, is

$$63360 \times \cdot 03617 \text{ or } 2292 \text{ lb. weight approximately.}$$

Hence the pressure required is

$$15 + 2292 \text{ or } 2307 \text{ lb. weight per square inch.}$$

It is clear from this last result that at great depths below the surface of the sea the pressure is enormous.

The various results might have been obtained by a direct application of the formula $p = \pi + \omega h$; they might also have been found in pounds weight per square foot or in other units.

The fact that the weight of 1 cubic centimetre of water is 1 gramme weight, simplifies the numerical work on the C.G.S. system very greatly, for if we work in grammes weight and centimetres we have on this system $\omega = 1$, and the formula for the pressure becomes $p = \pi + h$, or expressing π in terms of H_0 the height of the effective surface $p = H_0 + h$.

In this case the pressure in grammes weight per square centimetre and the "Head" in centimetres are numerically equal.

If the fluid considered be not water but some liquid of specific gravity σ, we have to find the weight of unit volume. On the English system this is $62 \cdot 5 \times \sigma$ pounds weight per cubic foot; on the C.G.S. system, it is σ grammes weight per cubic centimetre.

(2) *Assuming the atmospheric pressure to be* 1 *kilogramme weight per square centimetre, find the pressure* (i) *at a depth of* 500 *metres in fresh water*, (ii) *at the same depth in salt water of specific gravity* 1·026.

(i) In the fresh water the pressure is

$$1000 + 500 \times 100 \text{ or } 51000 \text{ grammes weight per square centimetre.}$$

(ii) In the salt water the pressure is

$$1000 + 500 \times 100 \times 1 \cdot 026 \text{ or } 52300 \text{ grammes weight per square centimetre.}$$

(3) *The specific gravity of mercury is* 13·6, *at what depth in mercury will the pressure be equal to that at* 500 *metres in sea water?*

Omitting the atmospheric pressure which affects both alike, the pressure due to 500 metres of sea water is 51300 grammes weight per square centimetre. Dividing this by the weight of a cubic centimetre in mercury in grammes weight we get the equivalent depth of mercury.

Now a cubic centimetre of mercury weighs 13·6 grammes weight.

Hence the depth required is

$$51300/13 \cdot 6 \text{ or } 3773 \text{ centimetres.}$$

(4) *The pressure in a steam boiler is* 12 *kilo-weight per square centimetre, find the head of mercury to which this is equivalent.*

Head required $= 12000/13 \cdot 6 = 882$ centimetres.

(5) *A layer of mercury* 25 *cm. deep, specific gravity* 13·6, *is covered by one of water of the same depth; above this there is a layer* 50 *cm. in depth of oil, specific gravity* ·9; *find the pressure at the bottom* (i) *in grammes weight per square centimetre*, (ii) *in centimetres of mercury, assuming the atmospheric pressure to be due to a head of* 76 *cm. of mercury.*

(i) The pressure at the surface is

$$76 \times 13 \cdot 6 \text{ grammes weight per square centimetre.}$$

The pressure due to the oil is

$$50 \times \cdot 9 \text{ grammes weight per square centimetre.}$$

The pressure due to the water is

$$25 \text{ grammes weight per square centimetre.}$$

The pressure due to the mercury is

$$25 \times 13 \cdot 6 \text{ grammes weight per square centimetre.}$$

Hence adding these together the pressure at the bottom is

$$1443 \cdot 6 \text{ grammes weight per square centimetre.}$$

(ii) To find the pressure in centimetres of mercury we must divide the value just found by the number of grammes in a cubic centimetre or 13·6.

We obtain as the height required the value 1443·6/13·6 or 106·15 cm., or we may reason thus:

The height of the column due to the atmosphere and layer of mercury combined is $76 + 25$ or 101 cm., that due to the oil is $50 \times \cdot 9/13 \cdot 6$ or 3·31, and that due to the water 25/13·6 or 1·84; adding these we obtain 106·15 cm. as before.

35. Calculation of Thrust.

PROPOSITION 13. *To find the thrust on a horizontal surface immersed in a fluid under gravity.*

Since the pressure in a fluid under gravity is the same at all points in a horizontal plane, the pressure over any horizontal surface is uniform. Thus the thrust on the surface is found by multiplying the pressure at each point by the area

of the surface. Hence if p be the pressure at each point of the surface and A its area the resultant thrust P is given by

$$P = Ap.$$

Moreover if the surface be immersed at a depth h, if π be the atmospheric pressure and ω the weight of a unit of volume of the fluid, then

$$p = \pi + \omega h.$$

Hence $$P = (\pi + \omega h) A.$$

Again, we have $$p = \frac{P}{A}.$$

Now in some cases P and A can be measured and hence p can be calculated.

36. Manometers. A manometer is an instrument for measuring fluid pressure. There are many forms of manometers. Some of the most common will be described.

(i) *The* U *tube Manometer or Siphon Gauge.*

This consists of a glass tube bent to the form of a U as shown at ABC, fig. 29.

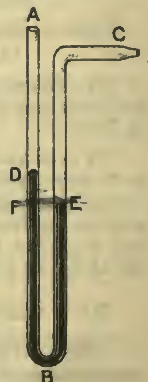


Fig. 29.

The lower part of the bend contains some liquid, say mercury, and the ends A and C are both open; the end C can be connected to the vessel in which it is desired to measure the pressure.

A scale is fixed alongside the tube so that the level of the mercury columns can be read.

When both ends are open the mercury stands at the same level in the two limbs; connect C to the vessel in which the pressure is to be measured; if this pressure be greater than that of the atmosphere the liquid in the limb BC is driven down, that in AB is raised, until the pressure, due to the difference of level of the mercury in the two limbs, together with the atmospheric pressure, balances the pressure on the surface of the mercury in BC.

Thus suppose D, E be the levels of the mercury columns in the two limbs. Draw EF horizontal to meet the column DB in F. Read the positions of E and D on the scale and thus find the height DE or DF; let it be h centimetres, let p be the pressure on the surface at E, π the atmospheric pressure, and ω the weight of a unit volume of the liquid in the manometer.

Now, since the pressures at two points in a given fluid in the same horizontal plane are equal, the pressures at E and F are equal; but the pressure at E is p, hence that at F is also p.

The pressure at F is the atmospheric pressure together with the weight of a column of fluid of height DF and unit cross section.

pressure = atmos press + (wt of unit of vol in the man × df)

Hence $$p = \pi + \omega h.$$

Thus the excess of the pressure at E over the atmospheric pressure is ωh. If h be in centimetres, ω in grammes weight per cubic centimetre, this difference of pressure will be in grammes weight per square centimetre. If mercury be the liquid used, the height h will measure the "head" in centimetres of mercury.

The choice of a liquid to be used will depend to some extent on the pressure to be measured; with a dense liquid like mercury a comparatively small head corresponds to a considerable pressure, hence, to measure pressures only slightly in excess of the atmospheric pressure, the head of mercury necessary would be small, and a small error in measuring it would mean a considerable error in the value of the pressure.

If a liquid of smaller specific gravity be used, the "head" necessary to measure a given pressure will be increased in the inverse ratio of the specific gravities, a given error in measurement will produce a proportionately less error in the result. Sulphuric acid, the specific gravity of which is about 1·842, is often used; water may be employed in some cases; it has however the disadvantage that it evaporates rapidly above the column EB, and the pressure due to the water vapour may cause error; sulphuric acid, on the other hand, absorbs water

quickly, its density therefore changes and this is a source of error.

This form of manometer is most useful to measure the pressure of a gas in a confined space; this pressure we have seen is the same throughout the mass. If it be used to measure that of a liquid we must remember that the pressure measured by the height of the manometer column is that at the surface E; if we wish to use it to measure the pressure of a liquid in a vessel connected to the manometer at C we must allow for the weight of the column of liquid between C and E.

EXPERIMENT 7. *To measure the pressure of the gas in the gas-pipes of the Laboratory.*

For this purpose connect the end C, Fig. 29, by means of a piece of india-rubber tubing with the gas-pipe; turn on the gas and read the difference in height between the two columns of liquid; the result gives the excess of pressure in the gas-pipes, over the atmospheric pressure, measured as a "head" of the liquid in the manometer.

If the specific gravity of the liquid be known, the pressure can be reduced to any other units. For this experiment water is a convenient liquid to use.

Example. *The pressure in a gas-holder exceeds the atmospheric pressure by* 10 *inches of mercury and the barometer*[1] *stands at* 30 *inches. If the specific gravity of mercury be* 13·6 *and that of sulphuric acid* 1·84, *determine the difference of level in a sulphuric acid gauge attached to the same gas-holder; find also the pressure on the walls of the gas-holder in lbs. weight to the square inch.*

The equivalent head of water is $10 \times 13{\cdot}6$ inches and of sulphuric acid it is $10 \times 13{\cdot}6/1{\cdot}84$ or 73·9 inches.

A cubic inch of water weighs 62·5/1728 or ·03617 lbs. weight.

Thus the pressure in lbs. weight per square inch is $136 \times {\cdot}03617$ or 4·92 lbs. weight.

We see from these results that whereas with a mercury gauge an error of ·1 inch in the height would mean an error of 1 per cent. in the pressure; with a water gauge it would mean an error of about 1 in 1300, and with acid gauge of about 1 in 740.

(ii) *Other forms of Siphon Gauge.*

In some cases, for measuring high pressures, the end A of

[1] The height of the barometer measures the atmospheric pressure. See Section 68.

the gauge is closed as in Fig. 30. When the pressure on the end E of the mercury column increases this end is driven down, and the air in AD is compressed. By measuring the extent of this compression the pressure of the air in AD can be found by Boyle's law (see Section 79), and hence the pressure at E can be obtained; this pressure is chiefly due to the compressed air; in most cases the difference in pressure due to the column of liquid DF will be small compared with that due to the compressed air and may be neglected; the pressures at E and D may be treated as the same.

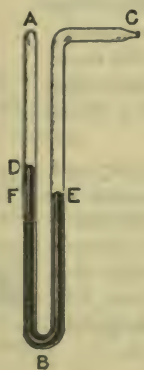


Fig. 30.

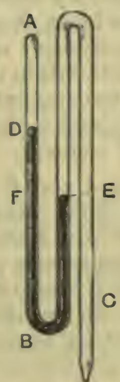


Fig. 31.

For measuring low pressures the tube AB is completely filled with mercury and its end is sealed up as in Fig. 31; then when the end C is open the atmospheric pressure forces the mercury up AB to the top of the tube, it stands at a lower level in the open tube; when the pressure in this second tube is sufficiently reduced the mercury in AB falls. Suppose that, as in Fig. 31, the surface of the mercury is at D, and in the other tube at E; draw EF horizontal, the pressure above the mercury at D is zero, and the pressure at E is equal to that at F.

The pressure at F is measured by the height of the column DF, which thus gives the pressure in the reservoir attached

to C. This form of gauge is commonly used with an air-pump. (See Section 98.)

37. Barometer Tube Gauge.

Another gauge for low pressures is shewn in Fig. 32. It consists of a vertical tube AB which dips at A into a vessel of mercury and communicates at the top with the vessel in which the pressure is to be measured. As the pressure in this vessel is reduced below that of the atmosphere the mercury rises in the tube[1]. Let its top surface be at D, a height h above the surface of the mercury at A in the reservoir, let E be a point[2] within the tube at the same level as A, π the atmospheric pressure, p the pressure at D, and ω the weight of a cubic centimetre of mercury.

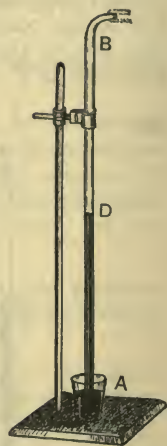


Fig. 32.

The points A and E in the mercury are at the same level, hence the pressures at these points are the same. But the pressure at A is the atmospheric pressure π. Hence the pressure at E is also π; now consider the column of mercury above E in the tube BA, the pressure at its top is p, and its height is h, hence the pressure at E is $p + \omega h$.

Thus $$\pi = p + \omega h.$$

Hence $$p = \pi - \omega h.$$

It should be noticed in all these cases that the size of the tube need not be taken into account.

38. The Safety-valve.

Another form of pressure gauge for high pressures is the safety-valve of a steam-boiler. A spherical or conical plug A, Fig. 33, fits accurately into a circular opening connected with the boiler, the pressure in the boiler tends to raise this plug, it is kept in position by a downward force applied from above.

[1] See Section 70.

[2] E is not shewn in the figure.

This downward force is usually exerted by a lever, the fulcrum of the lever is fixed, as shewn at C; the arm of the lever

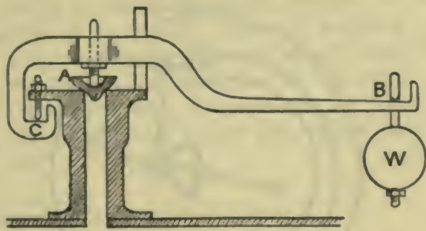


Fig. 33.

presses on the plug at A, the extremity of the arm either carries a weight W, or, if the boiler be not steady, is attached to a spiral spring.

Let W be the weight at B, or the downward pull registered by the spring. Let P be the upward thrust at A, p the pressure in the boiler, and a the radius of the orifice closed by the plug.

The effective area exposed to vertical fluid pressure is πa^2, hence the upward thrust is $p \cdot \pi a^2$.

Thus $$P = p \cdot \pi a^2.$$

But $P \times$ horizontal distance between C and $A = W \times$ horizontal distance between C and B.

And hence, $$p = \frac{W}{\pi a^2} \frac{\text{arm of lever}}{\text{distance of fulcrum from valve}}.$$

If p exceeds this value, the steam just begins to escape; the pressure therefore can be measured by adjusting either the weight W or the length of the arm CB until the steam just begins to blow off.

39. The Bourdon Gauge.

This consists of a tube AB, Fig. 34, of thin metal whose axis is bent into the form of the arc of a circle. One end of the tube A is closed, the other, B, communicates with the vessel in which the pressure is to be measured, the section of the tube is elliptical. Now when the pressure in the tube increases, this elliptical section tends to become circular. This causes the axis of the tube to uncurl slightly so that

the end A moves upwards; this slight motion of the tube is communicated to a pointer which moves over a circular scale

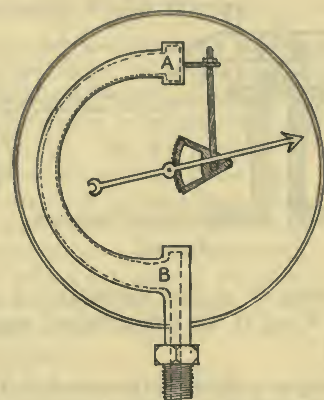


Fig. 34.

by means of a lever working a rack and pinion. The scale is graduated by the application of known pressures and then any pressure applied to the inside of the tube can be measured.

40. Experiments on Fluid Pressure.

We have already described in Section 29 observations on some effects of fluid pressure. Thus the column of water remains in the glass tube, Fig. 15, because the atmospheric pressure at each point of its base transmitted through the fluid is greater than the pressure due to the column of water above. The pressure, however, at the top of the column is less than the atmospheric pressure; hence the stretched india-rubber which covers the top of the tube in the same figure is bulged in.

We may use the apparatus shewn in Fig. 17 to verify some of the laws of fluid pressure. For let us suppose that the radius of the glass tube is a centimetres, the area of its cross

section is πa^2 square centimetres; hence if p be the fluid pressure on the glass covering its base, the upward thrust is $p\pi a^2$ and if h be the depth to which it is sunk, the value of p is $\varpi + \omega h$ where ϖ is the atmospheric pressure, ω the weight of a unit of volume of the liquid. There is however a downward thrust due to the atmosphere on the upper side of this piece of glass; if we neglect the thickness of the walls of the tube, and suppose therefore that the area of the plate subject to pressure is the same both above and below this downward thrust is $\varpi \,.\, \pi a^2$.

Hence the resultant upward thrust is

$$(p - \varpi)\, \pi a^2, \text{ or } \omega h \pi a^2.$$

Now let the weight of the glass be W, then when the glass just falls off we see that W must be just greater than $\omega h \pi a^2$; thus we find that when the glass falls off $W = \omega h \pi a^2$.

We can measure these quantities and thus verify the result. Such an experiment would prove that the difference between the pressure at a point in a fluid and the atmospheric pressure is proportional to the depth. To perform it we should hold the glass plate in position by means of the string and immerse it to some depth in the fluid, then loose the string and raise the tube gently until the glass plate just begins to separate from the tube and the water to enter below. Measure the depth to which the tube is immersed, weigh the plate and find the area of the cross section, we have all the quantities required to verify the formula.

Then repeat the experiment, varying the value of W by loading the plate with some convenient weight.

It is troublesome in this experiment to secure a good fit between the tube and the plate. The following experiments verify the laws more satisfactorily.

EXPERIMENT 8. *To shew that the thrust, on a horizontal surface in a liquid, is proportional to the depth and to the density of the liquid.*

You are given a cylindrical vessel containing water and a long cylindrical tube of brass[1]. This tube is closed at one end and will float in the water as shewn in Fig. 35. Marks are made on the brass tube at distances of 10, 20, 30 cm. from the bottom. Put shot into the tube until it floats in the water, say up to division 20. Then the tube is entirely supported by the thrust of the water on its base, for the pressures on the sides, being horizontal, cannot help to support it in any way; and hence the weight of the tube and shot is equal to the thrust exerted by the water on its base. Weigh the tube and shot. Let the weight be W_1. Place in the tube more shot until the tube sinks to division 30. The thrust on the base again balances the weight of the tube and shot. Weigh them again and let their weight be W_2. Also let P_1 and P_2 be the thrusts on the base in the two cases. Then we have seen that $P_1 = W_1$, $P_2 = W_2$. Now it will be found that the ratio of $W_1 : W_2 = 20 : 30$.

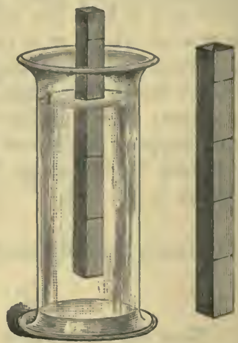


Fig. 35.

Therefore $P_1 : P_2 = 20 : 30$, or the thrusts on the base at two different depths are proportional to the depths.

The thrust at a given depth is also proportional to the density of the fluid. To prove this, place the brass tube in a second jar containing methylated spirit and determine the weight of tube and shot when the former sinks to 30 cm. Let this weight be W_2'. Then we wish to verify that the ratio

$$W_2' : W_2 = \text{density of methylated spirit} : \text{density of water},$$

or, since the density of water is 1 gramme per cubic centimetre, that

$$\text{Density of methylated spirit} = \frac{W_2'}{W_2} \text{ grammes per c.cm.}$$

[1] The tube may be circular, instead of square as in the figure, and may be made of a piece of drawn brass tubing; it may conveniently be 35 centimetres in height by from 2 to 3 centimetres in diameter, the diameter of the second tube used in Experiment 8 should be larger than this, say some 4 centimetres.

Look out the density of methylated spirit in the Table and verify this.

Repeat the experiment, using a fluid denser than water, such as a solution of salt.

Thus, *when a horizontal surface is subject to fluid pressure the resultant thrust on the surface is proportional to the depth and to the density of the fluid.*

Since the density is proportional to the weight of unit of volume, we may say that the thrust is proportional to the depth and to the weight of unit volume of the fluid.

EXPERIMENT 9. *To shew that the thrust on a horizontal surface immersed in a fluid is proportional to the area of the surface, and hence to find the pressure at any point in the fluid.*

Measure, by one of the methods given in Dynamics, Section 7, the area of the base of the tube used in the last experiment. If the section is circular it is simplest to measure the diameter of its base with the callipers; let it be d_1 cm. Then the area of the base is $\frac{1}{4}\pi d_1^2$ square centimetres. Determine the weight W_1 grammes of the tube and shot when it floats in water with its axis vertical and 20 centimetres immersed. Repeat the experiment, using a second tube of diameter d_2 centimetres so that the area of its base is $\frac{1}{4}\pi d_2^2$ square centimetres. Let W_2 be the weight of the tube and shot in this case. Then W_1 and W_2 measure the upward thrusts in the two cases and it will be found that

$$W_1 : W_2 = \tfrac{1}{4}\pi d_1^2 : \tfrac{1}{4}\pi d_2^2 = d_1^2 : d_2^2.$$

Thus the ratio of the two thrusts is the same as the ratio of the two areas.

41. Deductions from Experiments on Fluid Pressure.

Hence, combining the results of the two experiments 8 and 9,

The resultant thrust on a horizontal surface is proportional to the area of the surface, to its depth, and to the density of the fluid.

Now let p be the pressure at any point of the under side of the surface, and π the atmospheric pressure, which acts

directly on the upper side of this same surface, as well as indirectly, by transmission through the fluid on its under side. The areas subject to the pressures p and π are the same[1], and, for the first tube, are equal to $\frac{1}{4}\pi d_1^2$. Hence the resultant upward thrust for the first tube is

$$(p_1 - \pi)\,\tfrac{1}{4}\pi d_1^2,$$

and this is equal to W_1.

Thus

$$(p_1 - \pi)\,\tfrac{1}{4}\pi d_1^2 = W_1$$

$$p_1 = \pi + \frac{W_1}{\frac{1}{4}\pi d_1^2}.$$

Hence we have measured by experiment the pressure at a depth of 20 cm. in the water.

If the value of $W_1/\frac{1}{4}\pi d_1^2$ be calculated, W_1 being in grammes and d_1 in centimetres, it will be found to be 20; if the tube had been immersed to a depth h the value for $W/\frac{1}{4}\pi d^2$ would be h.

Thus we find that p the pressure at a point, at a depth h in water, is given by $p = (\pi + h)$ grammes weight per square centimetre.

Again, it follows from the experiment that if W' be the weight of the tube and its contents, when sunk to a depth h in a liquid in which the weight of unit of volume is ω, then $W' = \omega . W$.

But in this case if p is the pressure at the depth h in this liquid, we have as before

$$p = \pi + \frac{W'}{\frac{1}{4}\pi d^2} = \pi + \frac{\omega W}{\frac{1}{4}\pi d^2}.$$

And we have just seen that $W/\frac{1}{4}\pi d^2$ is equal to h.

Hence we obtain $p = (\pi + \omega h)$ grammes weight per square centimetre.

We have thus obtained from the experiments an expression for the pressure at a point in a fluid under gravity.

[1] The interior area at the bottom of the tube is of course less than the exterior by an amount depending on the thickness of the walls, but the atmospheric pressure acts vertically downwards at the top of the tube on an area exactly equal to this difference.

*42. Surfaces of Equal Density.

There are two theoretical propositions of importance with which we may conclude the chapter.

*PROPOSITION 14. *To shew that the densities at two points in a fluid at rest under gravity at the same depth are the same.*

Let A, B, Fig. 36, be two points at the same depth in a fluid at rest under gravity.

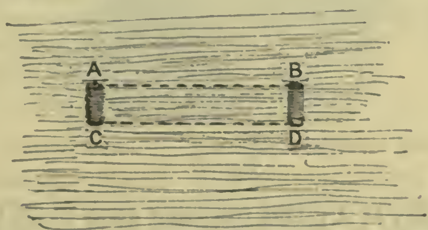


Fig. 36.

The pressures at A and B are equal, let each be p.

Take a point C a very short distance below A and a point D at the same short distance below B, so that $AC = BD$. Then C and D are at the same depth and the pressure at C is equal to that at D; let the value of this pressure be p'.

If C is very near to A we may treat the fluid as though its density between A and C were constant, and equal to its mean value between these points, let this mean value be ρ_1; similarly let ρ_2 be the mean value of the density between B and D.

Then we shall shew that $\rho_1 = \rho_2$.

For, by considering a small cylinder of fluid between A and C, we find $$p' = p + g\rho_1 AC.$$

While, by considering a cylinder of equal height between B and D, we have $$p' = p + g\rho_2 BD.$$

Hence $$\rho_1 AC = \rho_2 BD.$$

But $$AC = BD.$$

Therefore $\rho_1 = \rho_2$, or the density is the same at two points at the same depth.

Corollary. It follows from this that the common surface of two different liquids which do not mix is a horizontal plane; for, if not, let the surface lie as *EF*, Fig. 37. Then draw *AB* horizontal, so that *A* may be in one liquid, *B* in the other.

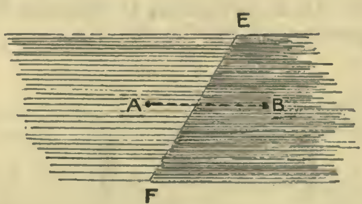


Fig. 37.

The density at *A* differs from that at *B*, which is contrary to the proposition just proved; thus *EF* cannot be as drawn, it must be horizontal.

Thus, in a fluid under gravity, surfaces of equal pressure are also surfaces of equal density.

*Proposition 15. *When two fluids which do not mix are in stable equilibrium, the upper fluid must be lighter than the lower.*

To prove this, imagine a closed tube filled with the two fluids, and having a stop-cock at the bottom by which it can be divided into two parts. If it be possible, let the heavier fluid fill the upper part of the tube. Then there will be equilibrium so long as the surfaces of separation *A*, *B*, Fig. 38, in the two branches of the tube are at the same level. Let the fluids now be displaced so that the surfaces of separation take the positions *C*, *D*, and suppose the stop-cock

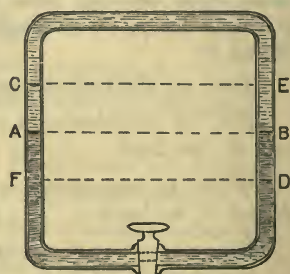


Fig. 38.

closed. Draw CE, DF horizontal to meet the fluid again in E and F. Let ω_1, ω_2 be the weights of unit volume of the upper and lower fluids respectively, and let h be the vertical distance between DF and EC.

Then, since C and E are points in the same fluid at the same level, the fluid pressures at these two points are equal. Let each be p. Let p_1 be the pressure at D, p_2 that at F.

Then, since the vertical distances CF and DE are equal, while the density of the fluid in DE is greater than that in FC, and the pressures at C and E are equal, the pressure at D is greater than that at F.

For we have
$$p_1 = p + \omega_1 h,$$
$$p_2 = p + \omega_2 h,$$
and ω_1 is greater than ω_2; hence p_1 is greater than p_2.

Now let the stop-cock be opened, then F and D are points at the same level in the same fluid; hence, for equilibrium, the pressure at these two points must be the same; but we have shewn that the pressure at D is greater than that at F, hence the equilibrium cannot be maintained and the column CE of the fluid will move. Moreover the surface C will ascend and E will descend; thus, if the two fluids be slightly disturbed from their original position, motion will be set up in such a way that the disturbance goes on increasing; the heavier fluid comes to the bottom, the original position was unstable.

If the lighter fluid had been at the top originally, then the same method of proof would have shewn that when the disturbance took place the pressure at D would be less than that at F, thus the column FD would, when the stop-cock is opened, move back to its original position; the equilibrium would be stable.

The proof will apply to a more general case when the two fluids instead of being contained in a tube are placed one above the other in an open vessel.

*43. Equilibrium of Two or more Fluids.

The proposition may be illustrated in various ways; if oil and water, which do not mix, be placed together in a beaker,

the oil rises to the top and floats on the water; it is lighter than the water, and the stable position of equilibrium is one with the oil on the top. It is possible, however, to arrange that the water should be above the oil.

Thus, take two small tumblers or wine-glasses of the same diameter, fill the one with water, the other with oil. Cover the top of the former with a thin card and invert it, holding the card so that no liquid escapes. Place it mouth downwards above the vessel containing the oil, the card separating the two liquids; on shifting or removing the card, so as to open a communication between the two, the oil will gradually rise into the upper vessel and the water sink into the lower one; the initial position with the water uppermost is unstable, and hence the transference occurs.

EXAMPLES.

1. Assuming the atmospheric pressure to be 1 kilo. wt. per square centimetre, find the pressure in water at the following depths:

25 cm., 1 metre, 1 mile, 5 kilometres;

and in mercury at the following depths:

1 cm., 1 metre, 25 metres, 1 kilometre.

2. The pressure at a certain point in a vessel of salt water is 35 lbs. wt. per square inch. Find the depth of the point, assuming the atmospheric pressure to be 15 lbs. wt. per square inch.

3. Determine the height of the mercury column which would produce the pressure given in Question 2.

4. Compare the pressures at equal depths in alcohol, carbon disulphide and water, neglecting the atmospheric pressure.

5. What head (1) of water, (2) of mercury is equivalent to a pressure of 14·5 lb. per square inch?

6. If the head of water above a point be 100 yards, what is the pressure at the point?

7. Find the heads (1) of water, (2) of mercury corresponding to pressures of 1 kilo wt. per square centimetre; 30 lbs. wt. per square inch; one million dynes per square centimetre.

8. Find the pressure in poundals per square foot due to a head of 30 inches of mercury.

9. What is the pressure at a depth of 60 fathoms below the surface in sea water?

10. A cylinder 3 feet in diameter is fitted with a piston and filled with water. A weight of 5 tons weight is placed on the piston, find the pressure in the water.

11. The head of water in a pipe communicating with a cylinder having a piston 2 feet in diameter is 200 feet. Find the force the piston can exert.

12. The pistons of a press are 2 inches and 10 inches in diameter; what is the pressure in the liquid when the small piston carries a load of 5 cwt. and what force can the large piston exert?

13. The pressure in a liquid at a depth of 60 inches is 30 lbs. per square inch; what is the thrust at a depth of 30 feet (1) on a square foot, (2) on a square yard?

14. The pressure in a well at a depth of 95 feet is four times that at the surface; find the pressure of the atmosphere per square inch of the surface.

15. Assuming the atmospheric pressure to be 1 kilo wt. per square centimetre, find the pressures at depths (1) of 76 cm., (2) of 380 cm. below the surface of mercury.

16. The pressure in a water-pipe at the base of a building is 40 lbs. wt. per square inch, on the roof it is 20 lbs. wt. per square inch; find the height of the roof.

17. A vessel in the form of a cube 1 metre in edge is filled with water; find the resultant thrust on its base.

18. What volume of mercury must be placed in the vessel to produce the same resultant thrust?

19. Express the pressure of the atmosphere in pounds weight per square foot when the height of the water barometer is 32 feet.

20. A vessel is partly filled with water and then olive oil is poured on until it forms a layer 6 inches deep; find the pressure per square inch at a point 8·5 inches below the surface of the oil, neglecting the atmospheric pressure.

21. A tube 20 feet long with one end open is filled with water and inverted over a vessel of water; what is the pressure in the water at the top of the column? The height of the water barometer is 33 ft.

22. A vertical tube is fixed alongside of a vessel and communicates with its bottom. The vessel contains mercury, water and olive oil, the depth of each being 10 inches. How high is the column of mercury in the tube?

23. Describe how the pressure at a point 12 inches below the surface of the water in a vessel may be measured by experiment, and how the experiment may be varied to shew on what this pressure depends.

24. The atmospheric pressure at the surface of a lake is 15 lbs. per square inch. Find at what depth the pressure is 45 lbs. per square inch, the weight of a cubic foot of water being taken to be 1000 ounces.

25. Describe an experiment to shew that the difference between the pressures at two points in a fluid at rest under gravity is proportional to the difference in their depths.

26. What will be the thrust on a square board whose side is 1 foot when sunk in water to the depth of 20 feet, the board being horizontal and the height of the barometer at the surface of the water 30 inches? The specific gravity of mercury is 13·59.

27. What is the pressure in lbs. per square inch at a point in mercury at a depth of 2 feet, the specific gravity of mercury being 13·59? The pressure of the atmosphere being neglected.

28. A layer of oil 25 cm. in depth and of specific gravity 0·82 floats on a quantity of water at the same depth. Find the difference between the pressure at the top surface of the oil and that at the bottom of the water.

29. A vertical cylinder is fitted with a smooth piston resting on water contained in the cylinder: from the side of the cylinder close to its base rises a vertical tube communicating with the cylinder, and therefore also containing water. Find the area of the piston so that for each pound placed upon it, the surface of the water in the tube may increase its vertical distance from the piston by 1 inch. [A cubic foot of water weighs 1000 ounces.]

30. A cylindrical barrel, the area of whose bottom inside the barrel is 5 square feet, has its axis vertical. A vertical pipe (area of its internal section 18 sq. inches) is screwed into a hole in the top of the barrel, and water poured in until the barrel is full, and also 7 inches of the pipe above the barrel. The uniform thickness of the top being 1 inch, find (i) the upward thrust of the water on the top of the barrel, (ii) what extra volume of water must be poured in, so that the upward thrust may be doubled? [A cubic foot of water weighs 1000 ounces.]

CHAPTER IV.

FLUID THRUST. CENTRE OF PRESSURE.

44. Thrust on a Horizontal Surface.

A value has already been found for the resultant thrust on a horizontal surface exposed to fluid pressure.

It is, if we omit the pressure on the free surface of the fluid, the weight of a column of the fluid having the horizontal surface for its base and the depth of that surface for its height. This follows from the expression found in Section 35, for, if A be the area and p the pressure at each point of the area, then, since the pressure is uniform, the resultant thrust P is given by the equation

$$P = Ap.$$

But if h is the height of the surface, and ω the weight of unit of volume of the fluid, then $p = \omega h$.

Hence $$P = A\omega h = \omega A h.$$

Now Ah is the volume of a cylinder, having the surface for its base, and the depth of the surface for its height, and $\omega A h$ is the weight of this cylinder if composed of the fluid.

The same result can be obtained by the graphical construction given in Section 22. We have seen there that the thrust on any small area may be represented by the weight of a small cylinder having the area for its base, and the depth of the area for its height.

Imagine now that the given surface is divided into a number of small areas and that such cylinders are drawn for each. These cylinders will all be of the same height; they will form a column with a flat top having the surface for its base; the height of the column gives the pressure at each point of the base; the weight of the column will be the total thrust on the surface.

If the column be of water, its weight in grammes is measured by its volume in cubic centimetres. Thus, to find the thrust, we require to calculate the volume of the column, which is done by multiplying the area of its base by its height.

45. Thrust on a Vertical Surface.

A similar method can be applied to find the thrust when the pressure is variable. For, in this case, if the pressure be p, the thrust on any very small area a is pa. Suppose now that the surface be placed in a horizontal position. Imagine it to be divided up into a very large number of small portions, each of area a. Suppose that on each of these a column of water be erected, in such a way that the height of the column in centimetres may be equal to the pressure p in grammes weight per square centimetre; the weight of a column will be pa grammes weight, and will measure the thrust on its base.

Hence the total weight of water above the surface is equal to the thrust, as in the previous case; since, however, the pressure is variable, the heights of the various columns are different, the tops of the columns no longer lie in a horizontal plane; if, however, the area a of each column be very small and if the pressure varies gradually from point to point, the tops of the columns will lie on a smooth surface; the volume of water bounded by this surface, the horizontal base and a series of vertical lines drawn from all points on the perimeter of the base can sometimes be found, and hence the total thrust can be found.

The direction of this total thrust passes through the centre of gravity of the volume of liquid, the point of the surface at which it acts is found therefore by drawing a line at right angles to the surface from the centre of gravity of the liquid. The point thus determined is known as the centre of pressure of the surface; its position can be found in various ways (see Section 46).

Example. *A rectangular plane surface is subject to fluid pressure. The pressure at any point is proportional to the distance of that point from one edge of the surface, find the thrust on the surface.*

Let $ABCD$, Fig. 39, represent the surface, and suppose the pressure at any point to be proportional to the distance of the point from the side AB.

Divide the surface up into a series of narrow strips by lines parallel to AB. Let PQ be one of these strips, all points on such a strip are equidistant from AB, and hence the pressure at all such points is the same. The pressure at any point on the strip PQ will be proportional to the distance AP. Let it be $k.AP$ where k is a constant, and let a be the breadth of the strip.

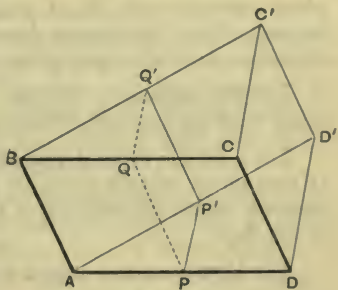


Fig. 39.

Draw PP' at right angles to the surface and equal to $k.AP$ and imagine a narrow vertical column of water of height PP' to rest on PQ. The weight of this column measures the thrust on PQ. Construct similar columns on the other strips into which the surface is divided; then, since the height of each column is proportional to the distance of its base from AB, the tops of the columns lie in a plane through AB, and when the breadth of each strip is made very narrow the volume of water whose weight represents the thrust will be bounded by the horizontal plane $ABCD$, by a second plane $ABC'D'$ through AB and by vertical planes through the lines AD, DC and CB.

The volume of this mass of water will be found by multiplying together the area of the triangle ADD', and the distance AB. The area of the triangle is

$$\tfrac{1}{2}AD.DD'.$$

But $DD'=k.AD$. Thus the area of the triangle is $\frac{1}{2}k.AD^2$. Hence the volume of the water is

$$\tfrac{1}{2}k.AD^2.AB.$$

If this volume be measured in cubic centimetres it will give the thrust on the plane $ABCD$ in grammes weight.

The question is really that of finding the thrust on a rectangular lock-gate, one side of which, AB, is in the surface of the fluid, when the effect of the atmospheric pressure is omitted; AD measures the depth of the gate and $AD^2.AB$ will be the area of the gate multiplied by its depth; if the liquid be water and the distances be measured in centimetres then k is unity.

Hence in these circumstances the thrust on a lock-gate in grammes weight is found by multiplying its area in square centimetres by half its depth in centimetres. Now the depth of the centre of gravity of the gate is half the depth of the gate. Hence in this case the thrust is the weight

of a column of the fluid whose base is the area under thrust, and height is the depth of the centre of gravity of that area. This result is a general one.

PROPOSITION 16. *To shew that the resultant thrust on any plane surface under fluid pressure is equal to the weight of a column of the fluid whose base is the area of the surface and whose height is the depth of the centre of gravity of the surface.*

Let the surface be divided into a number of elements, so small that we may treat the pressure as uniform over each.

Let a be the area of one of these elements, p the pressure at any point of this element, and let z be the depth[1] of the element below the surface, ω the weight of a unit of volume of the fluid.

The thrust on the area a is pa.

Since the surface is plane the thrusts on the various elements are all parallel; the resultant thrust is therefore the sum of the thrusts on the various elements.

Hence, if P be the resultant thrust, a_1, a_2, ... the areas of the various elements, then

$$P = p_1 a_1 + p_2 a_2 + \dots$$
$$= \Sigma (pa).$$

But $$p_1 = \omega z_1, \quad p_2 = \omega z_2, \dots.$$

Therefore $$P = \omega z_1 a_1 + \omega z_2 a_2 + \dots$$
$$= \omega \{z_1 a_1 + z_2 a_2 + \dots\}$$
$$= \omega \Sigma (za).$$

Now (*Statics*, § 38) we know that if $\bar{z}$ is the depth of the centre of gravity of a number of particles a_1, a_2, ... then

$$\bar{z} = \frac{a_1 z_1 + a_2 z_2 + \dots}{a_1 + a_2 + \dots} = \frac{\Sigma (za)}{\Sigma (a)}.$$

And in the case in point,

$$\Sigma (a) = \text{area of surface} = A.$$

Hence $$A\bar{z} = \Sigma (za).$$

Hence $$P = \omega \Sigma (za) = \omega A \bar{z}.$$

[1] If the pressure on the upper surface of the fluid is to be considered z must be measured from the effective surface, see Section 32.

Again, $A\bar{z}$ is a volume of fluid having the area A for its base and $\bar{z}$, the depth of the centre of gravity, for its height; while $\omega A\bar{z}$ is the weight of this volume. Thus the Proposition is proved.

Corollary. (i) The resultant thrust on a given plane surface does not depend on the inclination of the surface to the horizon, but solely upon its area and the depth of its centre of gravity. Thus, if the centre of gravity be fixed, the inclination of the surface to the horizon may be altered without altering the resultant thrust.

(ii) The resultant thrust does not depend upon the shape of the surface, but only upon its area, so long as the centre of gravity remains fixed in position; thus a plane surface of any shape, having a given area and its centre of gravity at a given depth, is subject to the same resultant thrust when immersed in a fluid.

46. Centre of Pressure.

DEFINITION. *The* Centre of Pressure *of any plane surface exposed to fluid pressure is the point of the surface at which the resultant thrust acts.*

Since the directions, in which the fluid pressure acts at each point of a plane surface, are parallel; the centre of pressure will be the point of application of the resultant of a system of parallel forces representing the thrusts on the elements of the surface; its position can therefore be found[1] by an application of the laws for determining the position of the resultant of a number of parallel forces.

In a few simple cases it is readily obtained by an application of the graphic method of Sections 22 and 45.

*PROPOSITION 17. *To find the centre of pressure of a rectangle with one side in the surface of the fluid.*

Let $ABCD$ be the rectangle, the side AB being in the surface of the fluid. The pressure at any point of the rectangle is proportional to the depth of the point; thus, if we divide the rectangle into a number of horizontal strips, the pressure is the same at each point of any given strip.

[1] Greaves, *Elementary Hydrostatics*, Chapter IV.

Draw CE and DF at right angles to the surface of the rectangle to represent the pressures at C and D, and join AF, FE, EB. Then, it follows from the graphical construction,

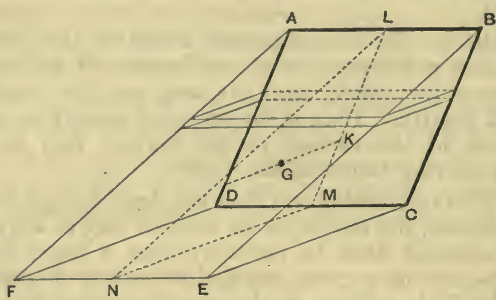


Fig. 40.

that the thrust on any element is represented by a column of the fluid bounded by lines drawn at right angles to the surface of the element to meet the plane $AFEB$; the resultant thrust is the weight of the wedge of fluid between the rectangle and $AFEB$, and acts, at right angles to the rectangle, through the centre of gravity of the wedge.

Now the wedge can be divided by planes parallel to BEC into a series of equal triangular laminæ, its centre of gravity therefore coincides with that of the triangle LMN, in which the wedge is cut by a vertical plane midway between AFD and BEC.

Let G be the centre of gravity and draw GK perpendicular to the plane of the rectangle. Then, since G is two-thirds of the way down the line joining L to the middle point of MN, it follows that LK is two-thirds of LM.

Thus, *The centre of pressure of the rectangle, with one side* AB *in the surface of a fluid, is a point at two-thirds of the depth of the bottom of the rectangle, midway between the sides* AD *and* BC.

Hence, if it were desired to balance the fluid pressure on one face of the rectangle by a single force, this force must be applied at the point K just found.

*Proposition 18. *To find the centre of pressure of a triangle with one side in the surface of the fluid.*

Consider a triangle ABC, Fig. 41, with one side BC in the surface. Let AD, drawn at right angles to the triangle, represent the pressure at the point A. Join BD, DC; then the weight of the tetrahedron $ABCD$ represents the thrust on the triangle; this thrust acts at right angles to the plane ABC, through G, the centre of gravity of the tetrahedron. Let AM bisect the base BC in M. Join DG and produce it to meet the triangle ABC in L, and let GK, drawn perpendicular to the triangle, meet it in K. Then K is the centre of pressure required. Moreover it is known from the properties of the tetrahedron that L is the centre of gravity of the triangle and that K and L both lie on the line AM.

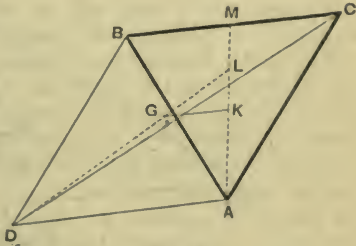


Fig. 41.

Also $\qquad DG = \frac{3}{4} LD$

and $\qquad AL = \frac{2}{3} AM.$

Hence, since AD and GK are parallel,

$$AK = \tfrac{3}{4} AL = \tfrac{3}{4} \times \tfrac{2}{3} . AM$$
$$= \tfrac{1}{2} AM.$$

Thus, *The centre of pressure of a triangle with its base in the surface of a fluid is situated on the line from the vertex bisecting the base and at half the depth of the vertex.*

*Proposition 19. *To find the centre of pressure of a triangle with one angular point in the surface of the fluid and its base horizontal.*

Let ABC, Fig. 42, be the triangle. Draw BD and CE at right angles to its plane, to represent the pressures at B and C. Join AD, AE and DE. Then the thrust on the triangle is the weight of the tetrahedron $ABCED$ acting at right angles to ABC through G, its centre of gravity.

Let GK be perpendicular on ABC; let L be the middle point of BC and M the centre of gravity of the base $BDEC$.

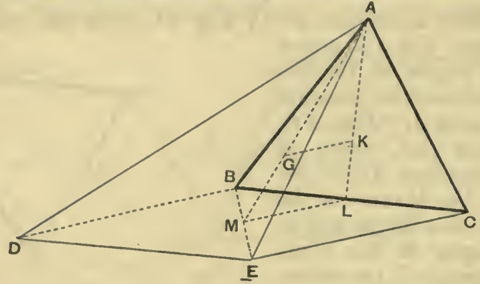


Fig. 42.

Then AGM and AKL are both straight lines, and AG is $\frac{3}{4}$ of AM.

Hence, since GK and ML are parallel, $AK = \frac{3}{4} AL$.

Thus, *If a triangle have its vertex in the surface and its base horizontal, the centre of pressure is on the line joining the vertex to the middle point of the base and at three-fourths the depth of this middle point.*

47. Thrust on the base of a vessel.

The expressions which have been found for the thrust on a plane surface exposed to uniform fluid pressure will apply to the case of liquid in a vessel with a flat bottom. It follows hence that

The Resultant Thrust *on the plane base of a vessel containing fluid does not depend on the shape of the rest of the vessel, but only on the area of the base and on the depth of the liquid.*

Thus in Fig. 43, (i), (ii), (iii) represent three vessels, the bases of which are equal in area. In (i) the sides are vertical, in (ii) they lean outwards so that the vessel is wider at the

(i) (ii) (iii)

Fig. 43.

top than at the bottom, in (iii) they lean inwards so that the reverse is the case. The depth of water in each vessel is the same. Under these circumstances the thrust on the bottom of each vessel is the same.

The pressure at each point of the base is the same; the thrust on any portion of the base is the weight of a column of liquid obtained by drawing vertical lines from all points of the boundary of that portion up to the level of the surface of the liquid. In (i) and (ii) such lines will lie wholly within the liquid. If the depth of the liquid be d centimetres there actually is a column of height d above any portion of the base.

For some parts of the vessel (iii) vertical lines drawn from the base will cut the sides. A column of liquid of height d does not exist above the whole of the base; the thrust on the bottom however is the same as though it did exist. Thus in Fig. 44, MNL represents a column cutting the side of the vessel in N, and CD the upper surface of the liquid produced in M.

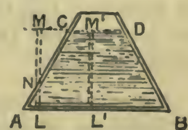


Fig. 44.

Then the thrust on the area of the base at L, on which the column stands, is the weight of the column ML, *not* that of NL. To prove this let L' be another point on the base from which a column $M'L'$ can be drawn to reach the surface CD in M'; then the pressure at L is equal to the pressure at L' since LL' is horizontal.

But the pressure at L' is the weight of a column of height $M'L'$ and unit cross section. Hence the pressure at L is the weight of a column of unit cross section and of height $M'L'$, which is equal to ML.

It sometimes appears paradoxical that the vertical thrust on any surface should be greater than the weight of the column of fluid immediately above it. In the case, however, of a column such as NL the fluid at N exerts an upward thrust on the surface at N; it is shewn, Section 48, that the vertical component of this thrust is the weight of the column MN, thus the surface of the vessel at N exerts a downward thrust on the top of the column NL whose vertical component is the weight of NM. This downward vertical thrust transmitted through the liquid together with the weight of the column NL, make up the resultant downward thrust on the base of the column at L, which is equal therefore to the weight of the column ML.

The **Resultant downward Thrust**, *on the base is found in all cases by drawing vertical lines from all points of the boundary of the base to meet the surface or surface produced, and is the weight of the column so formed.*

In Fig. 43 (i) this is equal to the weight of the liquid in the vessel; in (ii) the liquid extends outside the column, the thrust is less than the weight of liquid; in (iii) the column extends outside the liquid, the resultant vertical thrust is greater than the weight of the liquid. In all three cases if the areas of the bases and the depths of liquid be the same the resultant thrusts are equal.

In (i) the thrusts due to the sides of the vessel are entirely horizontal and do not affect the vertical thrust or help to support the liquid; in (ii) the thrusts due to the sides have an upward component, the weight of the liquid is in part supported by them; in (iii) the thrusts due to the sides act downwards and increase the thrust on the base beyond the weight of the contained liquid.

It should be clearly remembered in the above that we are dealing with the thrust on the base of the vessel containing the liquid, the force on the table on which the vessel stands is in all cases of course equal to the weight of the vessel together with the weight of the liquid.

In (ii) the downward thrust due to the weight of liquid above the sloping sides is transmitted through the sides to the supports carrying the vessel; the sides are forced more close to the base of the vessel.

In (iii) the upward thrust on the curved sides is transmitted through the sides to the base and just balances the excess of the downward thrust over the weight of the contained liquid; in (ii) the thrust on the sides would tend to close a small leak at the junction of the sides and the base; in (iii) it would tend to open such a leak.

EXPERIMENT 10. *To shew that the thrust on the base of a vessel filled with liquid depends only on the area of the base and on the depth of the liquid and not on the shape of the vessel.*

This is usually verified by an experiment due to Pascal and known as Pascal's Vases, Fig. 45. A number of vessels (or vases) differing in form can be screwed on to a stand. Each vessel is open at its base, and the area of the aperture is the same in each. When on the stand the vessels are closed

below by a moveable piece which is attached to one arm of a lever or balance. The other arm of the balance carries a scale-

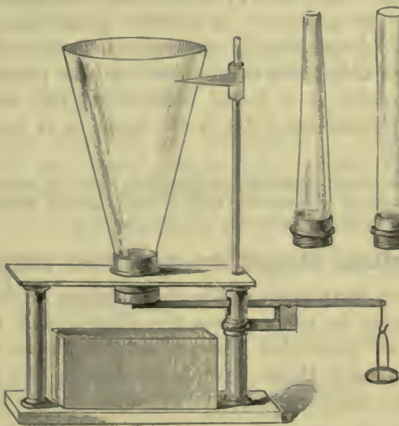


Fig. 45.

pan, into which weights can be placed. One of the vases is screwed on to the stand and weights are placed in the scale-pan. Water is then poured into the vessel until it just begins to raise the scale-pan and escape at the bottom; the level of the water is then noted by means of the pointer attached to the apparatus; the vessel is then removed and replaced by one of the others and the experiment is repeated; it is found that the scale-pan is again raised, and the water begins to escape when the level is the same as before.

Thus the thrust on the base depends on the depth of the water, and not on the shape of the vessel.

The experiment may be performed with more simple apparatus in the following manner:—

The end of a glass cylinder is ground flat and closed by a piece of flat glass supported by a string from the arm of a balance, the cylinder being held in a suitable stand. Weights are placed in the other pan and water poured in until it just begins to escape. The cylinder is then replaced by a glass tube

of different shape, a lamp chimney for example, having the same area of cross section, and the experiment is repeated. Instead of using a second vessel the cylinder may be closed some way below its top by a cork. A narrow glass tube is inserted through the cork projecting well above it and the water is poured through this tube, through which also passes the string carrying the plate, it is found that when the water escapes the level in the narrow tube is the same as it was previously in the cylinder.

In any of the above experiments however there is some difficulty in making the bottom fit sufficiently well to the various vessels to prevent leakage until the thrust reaches the proper amount.

48. Vertical thrust on a curved surface.

Hitherto we have been dealing with *plane* surfaces exposed to fluid pressure; in such a case the direction of the thrust is the same at all points of the surface, we have a system of parallel forces, and these have a resultant—the total thrust on the surface—which acts at right angles to the surface.

If we come to consider a curved surface the problem is more complex. The direction of the pressure is different at the various points of the surface; the thrusts therefore do not constitute a system of parallel forces and their resultant is more difficult to calculate. We may however resolve the thrust on each element of the surface into horizontal and vertical components, and calculate the resultant horizontal and vertical thrusts thus.

PROPOSITION 20. *To find the resultant vertical thrust on a surface exposed to fluid pressure.*

Let AB, Fig. 46, be such a surface and suppose we wish to

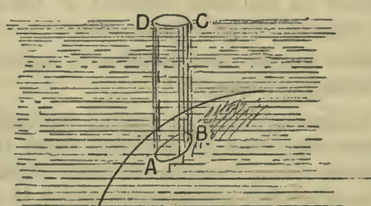


Fig. 46.

find the resultant vertical thrust on its upper side. From each point of the boundary of AB draw vertical lines to the surface, and consider the column of fluid $ABCD$ thus formed. It is in equilibrium under its own weight, the downward thrust on its upper surface, the thrust over the surface AB and the thrusts due to the surrounding portions of the fluid acting on its vertical faces.

Now these last are all horizontal. Thus the vertical component of the thrust due to AB, together with the thrust on the top surface, must balance the weight of the column of fluid.

Hence, Vertical component of thrust on AB

= weight of column $ABCD$

+ thrust on upper surface CD.

This result has been proved for the upper side of AB. If however the liquid is below AB, the upward thrust on the lower side of AB is equal and opposite to the downward thrust on the upper side of AB.

Hence the upward vertical thrust on the lower side of AB is equal to the weight of $ABCD$ together with the vertical thrust on CD.

In many problems we are not concerned with the effects of the thrust on the free surface of the liquid. In this case the vertical thrust acts through the centre of gravity of the column $ABCD$, and is equal to the weight of that column.

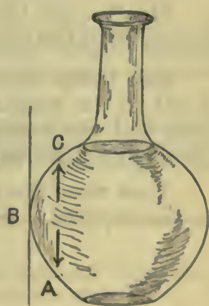


Fig. 47.

It may happen that the surface on which the thrust is required has a form such as ABC in Fig. 47.

Let B be a point at which the tangent to the surface is vertical, then the downward vertical thrust on AB and the upward vertical thrust on BC can both be calculated, the vertical thrust on ABC is the resultant of these two and is thus found.

49. Horizontal thrust on a curved surface.

PROPOSITION 21. *To find the resultant thrust in a given horizontal direction on a surface exposed to fluid pressure.*

Let AB, Fig. 48, be the surface. Draw lines from all points of the boundary of AB in the given horizontal direction. Let these lines cut in CD a vertical plane perpendicular to the given direction, and consider the equilibrium of the horizontal column of fluid thus formed.

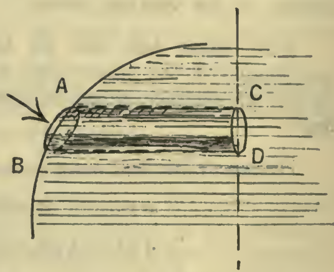


Fig. 48.

It is acted on by its weight, the thrust on AB, the thrust on CD, and the thrusts on its curved surface; these last thrusts are all at right angles to the lines bounding the column. The only forces then in the given direction, perpendicular that is to CD, are the thrust on CD, and the component, perpendicular to CD, of the thrust on AB. These two forces then must balance. Now it is this last component we desire to find.

Thus we see that the horizontal component, in the given direction, of the thrust on AB is equal to the thrust on CD.

This latter thrust has been shewn in Proposition 16 to be equal to the weight of a column of fluid having CD for its base and the depth of the centre of gravity of CD for its height; its direction passes through the centre of pressure of CD.

Thus when the thrust on CD can be found the component, in the required direction, of the thrust on AB is known.

EXAMPLES.

[For a Table of Specific Gravities see page 15.]

1. Shew that the force on the horizontal base of a vessel containing fluid does not depend on the shape of the vessel.

2. Determine the thrusts (*a*) on a square foot, (*b*) on a square mile of the earth's surface after the fall of ·95 inches of rain.

3. A conical vessel 10 inches high on a flat circular base 5 inches in radius is filled with water. Compare the vertical thrust on the base when the vertex is upwards with the vertical thrust on the curved surface when it is downwards.

4. A hollow cube with its base horizontal is filled with water, shew that the resultant thrust on each vertical face is half that on the base.

5. A hollow cylinder, whose section is a square 1 foot in edge, is filled with water. Find its depth if the thrust on each face is equal to that on the base, the height of the water barometer being 30 feet.

6. A closed cubical cistern is filled with water and communicates through a pipe with a second cubical cistern of eight times the volume. The second cistern is open and its surface is 30 feet above the base of the first. Compare the thrusts (*a*) on the bases, (*b*) on the vertical sides of the two, assuming the height of the water barometer to be 30 feet.

7. A tank in the form of a cube whose edge is 1 foot is half filled with water, half with olive oil. Find the thrust on a vertical face.

8. Find the thrust on a rectangular plate 12×8 inches immersed vertically in mercury with the short edge on the surface.

9. One of the vertical faces of a cubical tank 3 feet in edge is hinged about its upper edge. The tank is filled with water. What is the least force which must be applied to the face to retain it in its vertical position, and where must this force act?

10. The base of a rectangular tank the upper edges of which are 2 feet and 5 feet in length respectively is inclined so that when the tank is full it is 4 feet deep at one edge, 2 feet at the opposite edge. Find the resultant force on the base.

11. A tank 9 feet deep and 20 feet long is full of water; what is the total thrust on one side of the tank?

12. A rectangular tank is 12 feet long, 7 feet wide, and $2\frac{1}{2}$ feet deep; compare the thrusts on a side and on the bottom of the tank.

13. Shew how the pressure in a fluid varies with the depth; and find the resultant thrust on a fifty-foot length of the vertical retaining wall of a water reservoir 15 feet deep.

14. Determine the total thrust on one side of a rectangular vertical dock-gate 50 feet wide immersed in salt water to a depth of 25 feet, having given that a cubic foot of fresh water weighs 1000 ounces and that the specific gravity of sea water is 1·026.

15. The water on one side of the dock-gate in the previous question is fresh. What is its depth if the resultant thrusts on the two sides are equal?

16. An equilateral triangle is immersed in water, two of its angles being at a depth of 6 feet and the third at a depth of 9 feet. Find the force due to the water pressure on one side of the triangle.

17. A closed cubical cistern, each edge of which is 4 feet, is filled with water, and has a vertical pipe 10 feet high in its upper surface opening into it, also filled with water; if the atmospheric pressure be 14 lbs. per square inch, find the whole thrust on the base of the cistern.

18. A rectangular-shaped box is constructed with its ends (weightless) hinged to the base and capable of moving without friction between the sides, but so as to enable the box to contain water. The tops of these equal rectangular ends are connected by a piece of inelastic string so that, when water is poured into the box, they are inclined inwards and make equal angles with the vertical. Shew that the tension of the string varies as the cube of the depth of the water.

19. A tall conical champagne glass, the area of whose mouth is 2 square inches, is filled with wine of specific gravity$=1·296$, and the top is covered with a glass disc. The whole is then inverted and placed on a horizontal table so that no wine is spilt. If the weight of the champagne glass be just sufficient to prevent the wine from escaping, shew that the weight of the glass in ounces is equal to the depth of its conical contents in inches. [The volume of a cone is *one-third* of the volume of the cylinder with the same base and height.]

20. A vessel in the form of a portion of a cone is closed top and bottom by two circular plates, one being 3 inches and the other 5 inches in diameter, and filled with fluid. Compare the forces on the lower plate (1) when the larger one, (2) when the smaller one is at the bottom, and explain how it is that in the one case the pressure is greater and in the other less than the weight of the fluid.

CHAPTER V.

FLOATING BODIES.

50. Resultant vertical thrust on a body totally immersed.

In a previous Section (see Section 48) we have considered the resultant vertical thrust on any surface. When the body, whose surface is under consideration, is totally immersed a simpler expression can be found for the resultant thrust.

Let us consider in the first case a cylinder $ABCD$, Fig. 49, immersed with its axis vertical and its ends horizontal; the curved surface is then vertical; thus the pressure at each

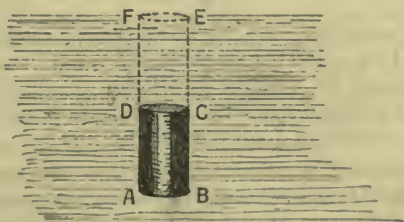


Fig. 49.

point of this surface is horizontal and cannot contribute to the vertical thrust.

Produce the vertical surface of the cylinder to meet the free surface of the liquid in EF. The resultant vertical thrust is the difference between the upward thrust on the bottom AB, and the downward thrust on the top CD. The upward thrust is equal to the weight of the column $ABEF$ together with the thrust on EF, the downward thrust is the weight of the column $CDFE$ together with the thrust on EF.

The difference between the two is the weight of a column *ABCD*, and acts upward through the centre of gravity of *ABCD*. Thus in this case the resultant vertical thrust on the cylinder is equal to the weight of the fluid displaced by the cylinder.

We shall shew in the next section that this result is true whatever be the shape of the body. We will however consider first another simple case. Let *ABC*, Fig. 50, be a body of spherical, or egg-shaped, form immersed in a fluid.

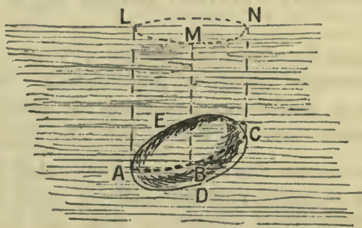


Fig. 50.

Imagine a vertical line *AL* to be drawn touching the body, and let this line move round the body always touching it, and always retaining its vertical position. It will thus trace out a cylindrical surface[1] touching the body in a curve *ABC*, and cutting the surface in a curve *LMN*.

The curve *ABC* divides the body into two parts; over the lower part *ABCD* the vertical pressure is everywhere upwards, the resultant vertical thrust on this part of the surface will thus be an upward force tending to raise the body, and it will be equal to the weight of the column of fluid *LADBM*. At points on the surface above *ABC* the pressure acts downwards, the resultant vertical thrust is thus downwards and is equal to the weight of the column *LAEBM*. The resultant vertical thrust on the whole body is the difference between these two and is equal to the difference between the weights of these two columns. This difference is the weight of the volume of fluid displaced by the body.

[1] We may imagine such a surface formed by rolling a sheet of paper so as to touch the body along a continuous curve.

The above method of proving this result, known as Archimedes' Principle, could be extended to more complex cases in which it was not possible to surround the body with a single vertical cylinder as in figure 50. The following proof will however apply to any case.

PROPOSITION 22. *To prove Archimedes' Principle, viz. that the resultant thrust on any body immersed in a fluid is equal to the weight of fluid displaced and acts vertically upwards through the centre of gravity of the fluid displaced.*

Let S, Fig. 51, be a body immersed in a fluid and held in position if necessary.

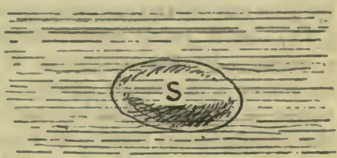


Fig. 51.

The fluid pressure at any point of the surface of S depends only on the depth of the point, and not on the nature of the surface on which the pressure acts. Imagine the body S to be removed, and the space it occupied filled with a quantity of the fluid, the rest of the fluid being undisturbed. The pressure at each point of the surface of this additional fluid is unaltered, and has the same value as when the solid was in its original position. Thus the resultant thrust on the solid is the same as that on the fluid which has replaced it. Now the fluid is in equilibrium under its own weight and the resultant thrust on its surface. This resultant thrust then must be equal to the weight of the fluid and must act vertically upwards through its centre of gravity.

Thus, *The resultant thrust on a solid immersed in a fluid is equal to the weight of fluid displaced, and acts vertically upwards through the centre of gravity of this displaced fluid.*

If the solid be not completely immersed, but cut the surface as ABC in Fig. 52, we must, in the same way, replace by fluid that portion of the solid which is below the surface.

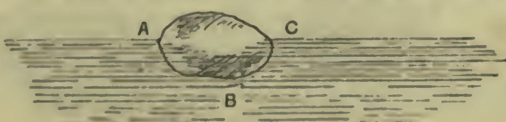


Fig. 52.

Definition. *The upward resultant thrust on an immersed solid is spoken of as the* Buoyancy of the Solid, *the centre of gravity of the fluid displaced is the* Centre of Buoyancy.

The results of the last few Articles may be verified by experiment.

Experiment 11. *To shew that the resultant upward thrust on an immersed body is equal to the weight of fluid displaced.*

(a) Take a body of which the volume can be found by measurement, a cube or cylinder suppose, and find its volume by the use of the calipers or in some other way (*Dynamics*, § 7); let it be V cubic centimetres. Weigh it in air with a balance, let the weight be W grammes weight. Immerse it in water and weigh again. In order to weigh a body in water it is suspended below the scale-pan of a balance or else the arrangement shewn in Figure 53 (a) is adopted. A stand or bridge rests on the floor of the balance case over one of the scale-pans which can swing freely below it; the supports of the scale-pan pass on either side of the bridge; the body to be weighed is suspended from a hook attached to the knife edge which carries the scale-pan. The vessel of fluid is placed on the

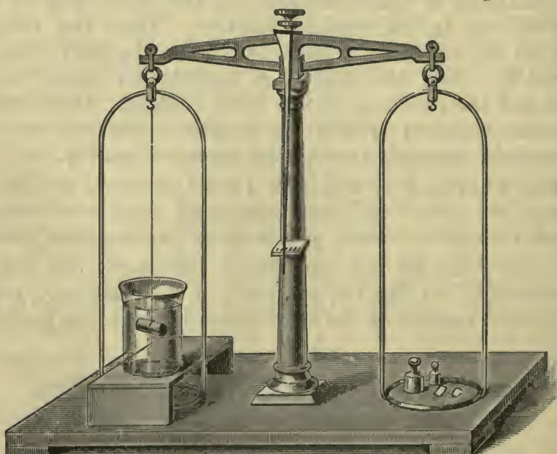


Fig. 53 *a*.

bridge and the body can be immersed in it. Let the weight of the body in water be W' grammes weight. The weight W' will be less than W because the body is partly supported by the fluid; the difference $W - W'$ will measure the resultant upward thrust due to the fluid. It will be found, if the fluid be water, that $W - W'$ is V grammes weight. Now we must remember that the weight of 1 c.cm. of water is 1 gramme weight, hence V c.cm. will weigh V grammes and we find thus that $W - W'$ is equal to the weight of V c.cm. of water. Hence the resultant upward thrust on the solid is equal to the weight of water displaced.

If a spring-balance graduated in pounds be used, its readings must be reduced to grammes by remembering that 1 lb. is approximately equal to 453 grammes.

(β) If the volume of the body cannot be found by direct measurement, it may be obtained by the displacement method described in *Dynamics*, Section 7, Experiment 4, and the experiment proceeded with as described above.

Experiment 12. *To determine the additional thrust on the*

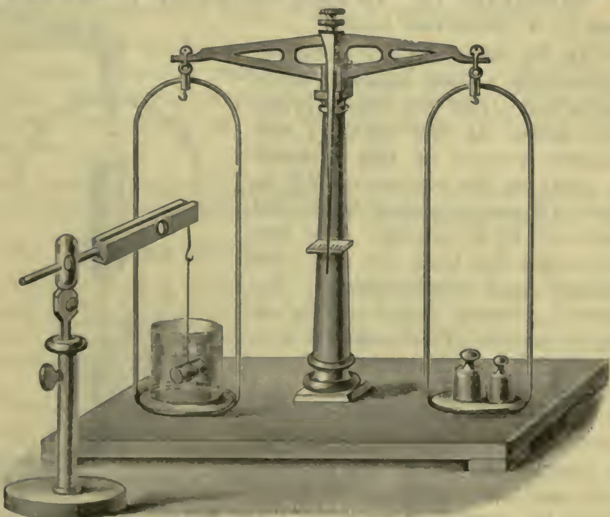


Fig. 53 *b*.

bottom of a vessel containing fluid produced by suspending a solid in the fluid.

This thrust must clearly be equal to the upward thrust of the fluid on the solid. To prove this, place a vessel containing water on the pan of a balance and counterpoise it. Take a body of known volume V c.cm. and suspend it in the water from a fixed support as shewn in Fig. 53 (b). The pan carrying the water is depressed; to restore equilibrium it will be found that V grammes weight must be placed in the other scale. Now this is the weight of water displaced by the solid.

The solid when placed in the water raises the level everywhere. Thus the pressure at each point of the base is increased; hence the thrust on the base is increased. Moreover if h is the rise of level and A the area of the water surface the additional thrust is the weight of a volume Ah of water. But since the level is raised by immersing the solid, Ah must be equal to the volume of the solid. Thus the additional thrust is a weight of water equal in volume to the solid.

Another experiment is the following.

EXPERIMENT 13. *To verify Archimedes' Principle for the case of a cylindrical body.*

In Fig. 53 (c) AB is a metal cylinder with a hook attached to its upper end. CD is a hollow cylinder closed at the bottom; the interior volume of this cylinder is equal to the volume of AB, so that AB can be placed inside CD and will fit it exactly. The two are suspended from one arm of a balance and counterpoised; the lower cylinder hangs inside an empty beaker. This beaker is then filled with water, the upward thrust on AB disturbs the balance and the arm carrying the cylinders rises. Water is then dropped from a pipette into the hollow cylinder CD, and its weight tends to restore the equilibrium; when CD is full it will be found that the

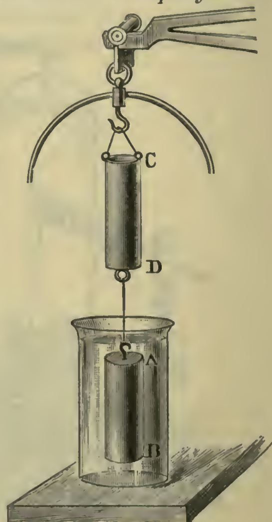


Fig. 53 (c).

balance beam is again horizontal. The upward thrust on AB is exactly balanced by the weight of water which fills CD, and the volume of this water is the same as the volume of AB; thus the buoyancy of AB is the weight of water which it displaces.

The experiment may be performed with some liquid other than water, the liquid which fills CD and that in which AB is immersed must of course be the same.

EXPERIMENT 14. *To find the volume of a body by weighing it in air and in water.*

Archimedes' Principle gives us a means of finding the volume of any body which sinks in water. Thus weigh the body in air, let it be W grammes weight. Weigh it in water, let it be W' grammes weight. Then the buoyancy or upward thrust is $W - W'$ grammes weight, and this is the weight of a mass of water equal in volume to the body. But the volume of $W - W'$ grammes of water is $W - W'$ cubic centimetres. Thus the volume of the body is $W - W'$ cubic centimetres.

If we are weighing in pounds we must, in order to find the volume, remember that a cubic foot of water weighs 62·32 pounds. Hence the volume of 1 lb. of water is 1/62·32 cubic feet and the volume of $W - W'$ pounds is $(W - W')/62·32$ cubic feet.

This method of finding the volume of a body is made use of in many of the experimental determinations of specific gravity. (See Chapter VI.)

51. Floating Bodies.

When a body is floating in a fluid partially immersed, the volume of liquid displaced is less than that of the body; the upward thrust is the weight of the liquid displaced, and acts vertically through the centre of gravity of the fluid displaced.

PROPOSITION 23. *To find the conditions of equilibrium of a body floating freely.*

Let ABC, Fig. 54, be the body and let G be its centre of gravity; let H be the centre of buoyancy, i.e. the centre of gravity of the fluid displaced. Then the body is in equilibrium under two vertical forces, viz. its own weight W acting

downwards at G, and the weight of fluid displaced W', acting upwards at H. It is necessary for equilibrium that these two

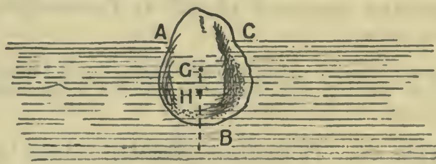


Fig. 54.

forces should be equal and opposite, for when two forces maintain a body in equilibrium they must be equal and their lines of action must lie in the same straight line. Hence the conditions of equilibrium are

(i) *The weight of the body is equal to the weight of fluid displaced.*

(ii) *The centre of gravity of the body is in the same vertical line as that of the fluid displaced.*

PROPOSITION 24. *A solid of given volume and density floats freely in a fluid of given density, to find the volume immersed.*

Let V, ρ be the volume and density of the solid, V' the volume immersed, ρ' the density of the fluid.

Then the weight of the solid is equal to the weight of fluid displaced; hence the mass of the solid is equal to the mass of fluid displaced.

But the mass of the solid is $V\rho$; the volume of fluid displaced is V', and its density is ρ', its mass therefore is $V'\rho'$.

Hence $$V\rho = V'\rho'.$$

Therefore $$V' = V\frac{\rho}{\rho'}.$$

If the fluid be water the ratio ρ/ρ' is the specific gravity of the solid. Hence, since ρ/ρ' is equal to V'/V we see that,

When a solid floats in water its specific gravity is the ratio of the volume immersed to the whole volume of the solid.

Corollary. If we can easily measure the value of V' we can use the result to compare the densities of different fluids. For we have

$$\rho' = \frac{V\rho}{V'}.$$

Now suppose the solid allowed to float in a fluid of density ρ'', and let V'' be the volume immersed.

Then $$\rho'' = \frac{V\rho}{V''}.$$

Hence $$\frac{\rho'}{\rho''} = \frac{V''}{V'}.$$

Hence *The densities of two fluids in which a given solid can float are inversely as the volumes immersed.*

We can prove this more briefly thus. When the solid floats, the mass of fluid displaced is equal to that of the solid. Hence the masses of the two fluids displaced are the same, hence their densities are inversely proportional to their volumes, i.e. to the volumes of the solid immersed.

This principle is made use of in the common Hydrometer. See Section 58.

Proposition 25. *To find the conditions of equilibrium of a solid immersed in a fluid and held by a string.*

(i) When the solid is totally immersed as in Fig. 55 the solid is acted on by three forces; its own weight, the buoyancy

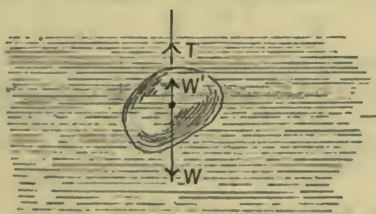


Fig. 55.

of the fluid, and the tension of the string. Since the solid is totally immersed, if we suppose it to be homogeneous, the centre of gravity of the fluid displaced coincides with that of the solid.

Thus the first two forces are vertical and act at the same point; the tension of the string is therefore vertical and its direction passes through the centre of gravity of the solid. Thus the point of attachment of the string is in the same vertical as the centre of gravity. Let W be the weight of the solid, W' that of the fluid displaced, the tension T is the difference between W and W'. If W is greater than W' the tension acts upwards, supporting the solid, and we have

$$T = W - W'.$$

In this case the solid is denser than the fluid. If W' is greater than W, or the fluid denser than the solid, then $T = W' - W$, and the string helps to keep the solid submerged.

*(ii) When the solid is not totally immersed. In this case the centre of gravity G and the centre of buoyancy H do not coincide. Let A, Fig. 56, be the point at which the

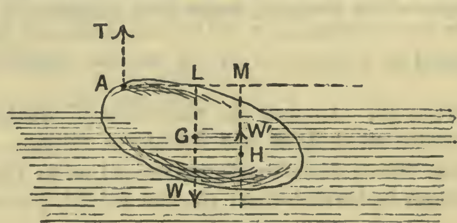


Fig. 56.

string is attached, then the forces W and W', acting at G and H respectively, are parallel, being both vertical in direction; hence the third force T is also parallel to them, the string is vertical, and, assuming that W is greater than W',

$$T = W - W'.$$

Again, let ALM meet in L and M the vertical lines through G and H respectively; then taking moments about A we have

$$W . AL = W' . AM.$$

This equation will determine the weight of the fluid which is displaced, and then the former equation will give T.

In case (i) it has been assumed that the solid is homogeneous so that its centre of gravity coincides with that of the fluid displaced; if this is not true the methods of (ii) must be applied to (i).

52. Buoyancy of the Air.

Experiments will be described in Section 66, which prove that air has weight. When therefore a body is weighed in air it is acted on by an upward thrust equal to the weight of the air displaced.

Hence the apparent weight of a body, when weighed in air, is not its true weight. The buoyancy of the air is illustrated by the ascent of a balloon. An air-tight bag of silk or some other light material is filled with hydrogen or coal gas or some other gas of less density than air, the weight of air displaced is then greater than the weight of the balloon, which can therefore rise and draw up with it a light car carrying passengers. The necessary condition is that the weight of air displaced should be greater than that of the balloon and its load.

In a fire-balloon the gas used is heated air, this is less dense than cold air; the bag is filled over a piece of sponge or cotton wick soaked in methylated spirits and ignited.

EXPERIMENT 15. *To illustrate the buoyancy of the air.*

In Fig. 57, A is a glass bulb with thin walls. A portion of the tube from which the bulb was blown is left attached, as

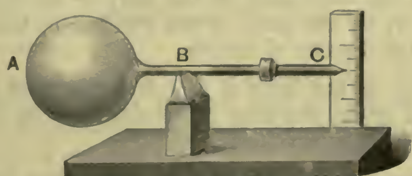


Fig. 57.

shewn at BC, being closed at the end C. By means of a small metallic counterpoise the whole is balanced with the stem horizontal on a knife edge at B.

The weight of air displaced by the bulb is much greater than that displaced by the tube BC and the counterpoise.

Since there is equilibrium, we have, taking moments round B,

Moment of bulb and contained air

− moment of air displaced by bulb

= moment of counterpoise and tube

− moment of air displaced by these.

The apparatus is then put inside the receiver of an air-pump (see Section 96), and the air is exhausted. As the exhaustion proceeds the bulb A sinks while the counterpoise rises. By withdrawing the air the upward thrust due to it is diminished; the buoyancy effect on the bulb is greater than that on the counterpoise, this is shewn by the ascent of the latter.

53. Corrections for weighing in air.

The buoyancy of the air affects the result of weighings made in air and a correction is in consequence needed. The correction is very small in most cases, but it is desirable to shew how to introduce it.

*Proposition 26. *To find the correction to the apparent weight of a body due to the buoyancy of the air.*

Let M be the true mass of the body to be weighed, ρ its density; let M_0 be the mass of the weights, ρ_0 their density, and let λ be the density of the air.

The volume of the body is M/ρ and the mass of air displaced by it is $M\lambda/\rho$, the mass of air displaced by the weights is $M_0\lambda/\rho_0$.

Now if the balance be true the difference between the weight of the body and the weight of air it displaces is equal to the difference between the weight of the "weights" and the weight of air they displace; and since the weights of these various bodies are proportional respectively to their masses

$$M - M\frac{\lambda}{\rho} = M_0 - M_0\frac{\lambda}{\rho_0}.$$

Therefore $$M = M_0 \left(\frac{1 - \frac{\lambda}{\rho_0}}{1 - \frac{\lambda}{\rho}} \right).$$

Now the value of λ varies with the pressure and temperature of the air, it is however very small compared with the density of solids or liquids, thus λ/ρ and λ/ρ_0 are very small quantities. If then we divide by the denominator $1 - \lambda/\rho$ and neglect λ^2/ρ^2, and higher terms which will be very small indeed, we find

$$M = M_0 \left\{ 1 - \frac{\lambda}{\rho_0} + \frac{\lambda}{\rho} \right\}.$$

For air under standard conditions λ is about ·0012 grammes per c.cm., while ρ_0 for brass weights it is about 8 grammes per c.cm., thus λ/ρ_0 is about ·00015.

The value of λ/ρ depends on the density of the body weighed: for water ρ is 1 gramme per c.cm. and λ/ρ is ·0012.

Thus the true mass M of a volume of water weighed with brass weights is given in terms of the apparent mass M_0 by the formula

$$\begin{aligned} M &= M_0 (1 - \cdot 00015 + \cdot 0012) \\ &= M_0 (1 + \cdot 00105). \end{aligned}$$

The correction in this case amounts to about ·1 per cent.

54. Experiments on floating bodies.

The following experiments illustrate some of the laws of floating bodies.

(i) The mass of an egg is greater than that of an equal volume of fresh water, it is less than that of an equal volume of a strong solution of salt in water; thus an egg will sink if placed in fresh water, it will float if placed in a strong solution of salt in water.

If a vessel is half filled with salt solution and then fresh water is carefully poured on to the top, the two liquids mix where they come into contact, forming layers of variable density; the egg placed in the fresh water will sink, but after oscillating up and down for some time will come to rest in a position in which the mass of fluid displaced is equal to that of the egg.

(ii) *The Cartesian diver.*

A small glass bulb, Fig. 58, has an opening in its lower side; to the bulb is attached a small counterpoise, the weight of which is adjusted so that the whole just floats in water with some air in the bulb—the counterpoise often takes the form of a glass figure; hence the name of the apparatus. The water is contained in a tall jar and its top is closed with a piece of india-rubber. On pressing the india-rubber down the pressure of the air above the water is increased; this pressure is transmitted to the air in the bulb, which contracts in volume. More water enters the bulb, the weight of the bulb with its contents becomes greater than the weight of water which it displaces. Hence the diver sinks. When the pressure at the top of the vessel is relieved, the air in the bulb expands and the diver rises unless the vessel is too deep. If the vessel exceed a certain depth the pressure at the bottom, due to the water, may be so great even when the air-pressure on the surface is relieved, that the air in the bulb cannot expand sufficiently to again allow the diver to rise. By placing the vessel under the receiver of an air-pump and exhausting, it may again be brought to the top.

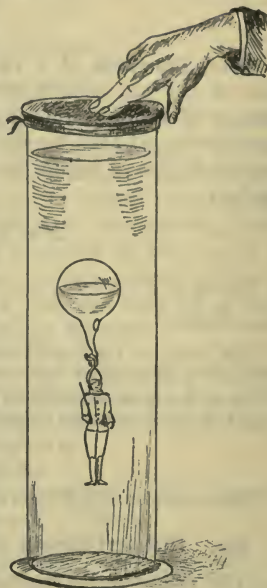


Fig. 58.

*55. Stability of equilibrium of a floating body.

So far we have dealt only with the conditions of equilibrium; the problem whether the equilibrium is stable or not remains to be discussed.

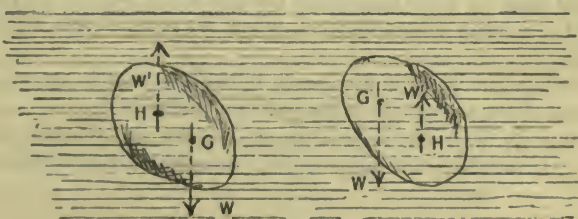
*(i) *When the body is totally immersed and just floats.*

The centre of buoyancy is the centre of gravity of the fluid displaced; if the body and the fluid be homogeneous it coincides with the centre of gravity of the body. In this case the two forces which act on the body pass through this point, and any position is one of equilibrium. The equilibrium therefore is neutral.

But if the centre of gravity and centre of buoyancy do not coincide, because, for example, either the body or the fluid is not homogeneous, various cases arise.

Suppose in the first case the body is not homogeneous. It may for example consist of a piece of wood with some lead fastened to it just sufficient in amount to allow it to float, or of a glass bulb and stem with some mercury in the bulb. Then we must distinguish between displacements in which the body moves without rotation—for these the equilibrium is neutral—and displacements in which the body is turned about some axis.

Fig. 59 (i), (ii) represent two cases which may occur; in each G is the centre of gravity, H the centre of buoyancy, in



(i) Fig. 59. (ii)

(i) G is below H, in (ii) it is above, and in each case the body has been displaced from its equilibrium position. The weight of the body acts downwards at G and the buoyancy acts upwards at H; these forces are equal and thus constitute a couple. In (i) the couple tends to right the body, the equilibrium is stable; in (ii) it tends to increase the displacement, the equilibrium is unstable. For stable equilibrium the centre of gravity must be as low as possible.

Or take again the case of the egg floating in a solution of

salt and water of variable density; the lower layers are the denser; if the egg is depressed it displaces a volume of fluid greater in weight than itself, hence it rises again; while if it is raised the fluid displaced becomes less in weight than the egg and it sinks back to its original position, thus the equilibrium is stable. With the diver on the other hand the reverse is the case; when it begins to sink it continues to sink until the air pressure on the surface is reduced, or the bottom is reached.

*(ii) *When the body is only partially immersed.*

The circumstances are now somewhat different. In the first place, such a body, if it can float at all, is in stable equilibrium for vertical displacements; if it be pushed down the buoyancy is increased, and it rises again; if it be raised out of the water the buoyancy is decreased, and the body sinks back.

But, though this is the case, the equilibrium for rotational motion may be either stable or unstable. A thin flat board *could* be made to float with its flat faces vertical and its edge downwards; in this position, however, it would be unstable and would tend to turn over until the flat sides became horizontal and the narrow edges vertical, when its equilibrium would be stable.

When the body is displaced, it is clear that in general the centre of buoyancy shifts its position in the body.

Let G, Fig. 60, represent the centre of gravity of the body, H its centre of buoyancy in the undisturbed equilibrium

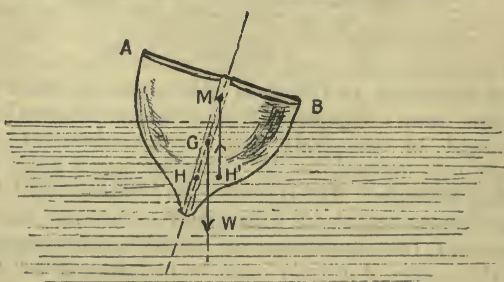


Fig. 60.

position, H' the new position of the centre of buoyancy when the body is displaced. Let us suppose that the form of the body is such that H, G and H' lie in a vertical plane, and also that the motion is such that the volume of water displaced remains unaltered, so that the buoyancy is unchanged[1]. Draw $H'M$ vertical through H', it will meet the line HG; let M be the point of intersection, then the nature of the equilibrium will depend on the position of M. The buoyancy now acts vertically upwards through M, the weight vertically downwards through G, and these two forces constitute a couple. If M be above G the couple tends to restore the body to its original position, if M be below G the couple tends to displace the body further.

The position of H' and therefore of M depends on the shape of the body. If the displacement be very small the point M is known as the Metacentre, and the condition of equilibrium for small displacements is that the metacentre should be above the centre of gravity.

*DEFINITION. *Imagine a floating body to be displaced through a small angle about a horizontal axis in such a way that the volume of fluid displaced may remain unchanged, and suppose further, that the displacement is such that the vertical line through the new position of the centre of buoyancy will meet the line joining the centre of gravity to the original position of the centre of buoyancy. The point of intersection of these two lines when the displacement is very small is the* **Metacentre.**

Thus we see that for small displacements the stability of a boat or other floating body depends on the position of the metacentre relative to the centre of gravity; for a boat of given shape and given weight it is important to keep the metacentre as low as possible. When the displacements to be dealt with may, as in the case of a ship, be considerable, other points besides the position of the metacentre must be attended to.

Examples. (1) *A piece of wood weighing 24 grammes floats in water with 2/3 of its volume immersed. Find the density and the volume of the wood.*

(i) Since 2/3 of the volume is immersed and the weight of the wood is equal to that of the water displaced, each cubic centimetre of the

[1] This condition is satisfied in the case, for example, of a ship which is made to heel either by the wind or through shifting some of the cargo.

wood weighs as much as 2/3 of a cubic centimetre of water or 2/3 of a gramme.

Thus the density of the wood is 2/3 of a gramme per cubic centimetre.

The volume of the wood is its mass divided by its density or $24/\frac{2}{3}$ c.cm. and this comes to 36 c.cm.

Thus the volume is 36 c.cm.

(ii) Otherwise thus. Let V be the volume in c.cm.

Then $\frac{2}{3}V$ is the volume of water displaced, and $\frac{2}{3}V$ grammes is the mass of water displaced; this is equal to the mass of the body.

Hence
$$\tfrac{2}{3}V=24,$$
$$V=36 \text{ c.cm.}$$

Also Density = mass/volume = 24/36 = $\frac{2}{3}$ grammes per c.cm.

(2) *A piece of wood of specific gravity* ·6 *is floating in oil of specific gravity* ·85, *what fraction of its volume is immersed?*

Since the volume of oil displaced is equal in weight to the wood, the two volumes are inversely as the specific gravities.

Thus fraction of whole immersed $= \frac{60}{85} = \cdot 7$.

(3) *A body whose volume is* 30 *c.cm. and specific gravity* 1·5 *is placed in a vessel and just covered with water. What thrust does it exert on the bottom?*

The weight of the body is 45 grammes, the weight of water displaced is 30 grammes, hence the thrust on the vessel due to the body is 15 grammes weight.

(4) *The mass of a balloon and its car is* 3000 *lbs., the mass of air displaced is* 3400 *lbs., with what acceleration does the balloon rise?*

The resultant upward force is

(3400 − 3000) or 400 lbs. weight.

This is equal to 400 × 32 poundals.

The mass moved is 3000 pounds.

Hence the acceleration is 400 × 32/3000 or about 4·26 feet per second.

(5) *A vessel containing water is placed on the pan of an upright spring balance and a body suspended from a second spring balance is immersed in the water. Examine how the readings of the two balances are altered.*

The reading of the second balance is reduced by the buoyancy of the water, acting upwards on the suspended body; the reading of the first balance is increased by the same amount, for the upward thrust of the water on the body is just equal to the additional downward thrust on the bottom of the vessel, due to the immersion of the body in the water.

Thus the sum of the two readings is unchanged, and this clearly must be so for the total mass supported by the two balances is not changed. See Experiments 11 and 12.

(6) *The specific gravity of sea water is* 1·028 *and of ice* ·918. *What fraction of the volume of an iceberg floats out of water?*

The weight of the iceberg is equal to the weight of water displaced and the weight of any body is proportional to the product of its volume and its specific gravity. Hence,

Volume of sea water displaced × 1·028 = volume of iceberg × ·918.

Hence $$\frac{\text{volume immersed}}{\text{whole volume}} = \frac{\cdot 918}{1\cdot 028}.$$

Therefore

$$\frac{\text{volume above water}}{\text{whole volume}} = \frac{1028 - 918}{1028} = \frac{110}{1028} = \cdot 107.$$

(7) *A body weighing* 10 *lbs. floats in a liquid with* 1/3 *of its volume above the surface. What weight must be placed on the body in order just to sink it?*

The specific gravity of the body is 2/3, hence the weight of water displaced by the body when just immersed is $\frac{3}{2}$ of 10 lbs. or 15 lbs. Hence the body will just float totally immersed if a weight of 5 lbs. be placed on it.

(8) *A rectangular block of boxwood* 10 *cm. in depth and of specific gravity* ·9, *is floating in water with its upper surface horizontal. Oil of specific gravity* ·6 *is poured on to the water, shew that, neglecting the buoyancy of the air, the wood will rise through* 1·5 *cm.*

Initially 9/10 of the wood is immersed, thus the height out of the water is 1 cm.; the weight of the wood is equal to the weight of water displaced.

When the oil is poured on, the weight of the wood is equal to the weight of water displaced together with the weight of oil displaced. Thus the weight of water displaced must be less than before, the wood therefore rises; let it rise h cm. and let A square cm. be the area of its upper surface.

The volume of the wood is $10\,A$ c.cm., that of the water displaced is $(9-h)\,A$ c.cm., and of oil $(1+h)\,A$ c.cm.

Equating then the weight of the wood to the weights of oil and water we have

$$\cdot 9 \times 10\,A = (9-h)\,A + \cdot 6\,(1+h)\,A.$$

Hence $4h = 6$ or $h = 1\cdot 5$ cm.

Thus the wood rises 1·5 cm., remaining with 7·5 cm. in the water and 2·5 cm. in the oil.

(9) *A cylindrical rod weighted at one end floats in water, determine the conditions of stable equilibrium. If the rod can float with half its length immersed and at any angle to the horizon, shew that the weight which is added is equal to that of the rod.*

Let W be the weight of the rod, $2a$ its length, W_1 the weight added, and w the weight of a unit volume of water, l the length immersed, and a

the area of the cross section, and h the distance from the bottom of the centre of gravity of the rod and weight.

Then the weight of water displaced is $la\omega$, hence

$$W + W_1 = la\omega,$$

also taking moments about the bottom of the rod

$$h(W + W_1) = aW.$$

The centre of buoyancy which is at a distance $\frac{1}{2}l$ from the bottom must be above the centre of gravity.

Hence $\frac{1}{2}l$ is greater than h.

Thus l is greater than $2h$, or

$$\frac{W + W_1}{a\omega} > \frac{2aW}{W + W_1},$$

or

$$\frac{(W + W_1)^2}{W} > 2aa\omega.$$

Now $2aa\omega$ is the weight of a volume of water equal to the volume of the rod, let this be W', then the condition for stability is that

$$(W + W_1)^2 \text{ is greater than } W \,.\, W'.$$

If the rod floats half immersed then $l = a$, while if any position is one of equilibrium, the centre of gravity coincides with the centre of buoyancy, hence

$$h = \tfrac{1}{2}l = \tfrac{1}{2}a.$$

Hence $$W + W_1 = 2W \text{ or } W = W_1.$$

EXAMPLES.

FLOATING BODIES.

[For a Table of Specific Gravities see p. 15.]

1. A piece of oak 35 c. inches in volume floats in water, what volume of water does it displace?

2. What is the weight in water of 1 kilogramme of iron?

3. Ten kilogrammes of cork are totally immersed in water. What is the resultant thrust of the water, and what will be the acceleration with which the cork will rise if let go?

4. Find the resultant upward thrust on the following bodies when totally immersed:

(i) 100 c.c. of iron in water.

(ii) 250 grammes of copper in salt water.

(iii) 500 c.c. lead in sulphuric acid.

5. A lump of iron floats in mercury. What fraction of its volume is immersed?

6. A lump of iron is supported in water by cork so that $\frac{1}{4}$ of its volume is out of the water. Compare the volumes of the cork and iron.

7. What is the apparent weight of 25 grammes of iron when weighed in alcohol?

8. Find the weight of water displaced by 56 grammes of copper.

9. An iceberg floats with 2000 c. feet above the surface of salt water; find its volume.

10. A piece of cork floats in water with 50 c. inches above the water; find its volume.

11. A body weighing 30 grammes floats in water with $\frac{2}{3}$ of the volume submerged; find its volume.

12. A cylinder floats, with its axis vertical, totally immersed in water covered with a layer of oil. If $\frac{2}{3}$ of the cylinder be in the water, find its specific gravity.

13. Find the weight of a glass ball 2 inches in diameter (1) in air, (2) in water, (3) in alcohol. [Sp. gr. glass 3·6.]

14. A vessel of water is placed on one pan of a balance and counterpoised. If 35 grammes of lead, supported by a string are immersed in the water, what additional weights are required to restore the balance?

15. Two bodies whose specific gravities are 2·5 and 7·5 balance when each is weighed under water; find the ratio of their weights.

16. A man weighing 10 stone floats with 5 cubic inches out of the water; find his mean specific gravity and his volume.

17. A piece of oak $\frac{1}{3}$ immersed in water is supported by a string. What portion of the weight is carried by the string?

18. A lump of iron floats totally immersed partly in mercury and partly in water. What volume is there in each liquid?

19. Find the specific gravities of bodies which float with the following volumes above and below the surface of water respectively:

(1)	Wood 21 : 24,	Ice 21 : 237,
	Cork 3 : 1,	Oak 1 : 3.

20. A ship weighing 1000 tons passes from fresh to salt water, if the area of a section at the water line be 15000 square feet and the sides where they cut the water be vertical, how much will she rise?

21. A piece of iron weighing 275 grammes floats in mercury with $\frac{5}{9}$ of its volume immersed. Determine the volume and density of the iron.

22. How much water will overflow from the edges of a cup just full of water when a cork 2 cubic inches in volume is gently placed in it so as to float?

23. A cylindrical cork 4 inches long is to be loaded with iron of the same section so as just to float. How long must the iron load be?

24. A cone of a certain material floats point downwards with $\frac{2}{3}$ of its axis immersed; find its specific gravity.

25. A vessel contains water and mercury. A cube of iron, 5 cm. along each edge, is in equilibrium in the liquids, with its faces vertical and horizontal. Determine how much of it is in each liquid, the densities of iron and mercury being 7·7 and 13·6 respectively.

26. A piece of wood is floating on water, and oil is then poured on to the water. How is the volume of wood in the water affected?

27. A cubic metre of wood floats in water with $\frac{3}{4}$ths of its volume immersed. Calculate the depth to which it would sink in a liquid of specific gravity ·8.

28. A block of wood 10 lbs. in weight floats in water with two-thirds of its volume immersed. What force will be required just to sink it? Also what weight of a metal (specific gravity 5) must be placed on it so that both metal and wood may just be immersed?

29. A closed cubical vessel with walls one inch in thickness is to be made of metal whose specific gravity is $\frac{27}{19}$. Shew that in order that the vessel may float in water its internal dimensions must be at least 64 cubic inches.

30. A block of hard wood, 5 feet long, 6 inches wide and 4 inches thick weighs 52 lbs. Determine whether it will float (1) in ordinary water, (2) in sea water, the specific gravity of which is 1·026. If in either case it does float, how much of it will project above the surface?

31. A body whose specific gravity is 3 and whose weight is 6 lbs. is supported by a string with half its volume immersed in water. What is the tension of the string, the density of water being unity?

32. A small hole drilled at one end of a thin uniform rod is filled with some much denser material. It is observed that the rod can float in water half immersed and inclined at any angle to the vertical. Shew that the specific gravity of the rod is $\frac{1}{4}$.

33. A beaker of water is placed on the pan of a balance and counterpoised, a piece of glass suspended from a separate support is immersed in the water and it is found that 20 grammes have to be added to the counterpoise to restore equilibrium. Explain this and calculate the volume of the glass.

34. Account for the ascent of a balloon.

How would the lifting power of the balloon be altered if the atmospheric pressure be diminished?

35. Why does a ship rise when it goes out of a fresh-water river into the open ocean?

36. The specific gravity of ice is 0·92, that of sea water is 1·025. What depth of water will be required to float a cubical iceberg whose side is 100 feet?

37. A piece of iron (specific gravity 7·2) is covered with wax (specific gravity 0·96) and the whole just floats in water; its mass is 36 grammes; find the mass of the iron and the wax respectively.

38. A cubic foot of ice at the freezing point, one of whose edges is one foot long, floats in ice-cold water, but so that it is capable only of rotation about one edge which is hinged in the surface of the water: a weight of 42 ounces placed on the top of the cube of ice 10 inches from its hinged end just immerses the ice in the water. Find the specific gravity of the ice.

39. A large stone is held suspended under water by a rope. Explain why the load on the rope is less under these conditions than if the stone were suspended in air. If an addition of 10 lbs. to the load on the rope will break the rope, how much of the stone may be raised out of the water before the rope breaks?

CHAPTER VI.

MEASUREMENT OF SPECIFIC GRAVITIES.

56. General Considerations.

In many methods of determining specific gravities the fact that the resultant upward thrust on a body immersed in a liquid is equal to the weight of liquid displaced is made use of, for by this means we obtain the weight of a volume of liquid equal in volume to the body, and the specific gravity has been defined, Section 6, as the ratio of the weight of the body to the weight of an equal volume of some standard substance, usually water.

The procedure generally adopted is to weigh the body in air and then in water; the difference between the two gives the weight of water displaced, that is, the weight of an equal volume of water, and the ratio of the weight in air to this difference, is the specific gravity.

The **Hydrostatic Balance** and various forms of **Hydrometers** give examples of this method.

57. Hydrostatic Balance.

This is an ordinary balance arranged so that a body suspended from one end of the beam may be easily weighed in water.

In some forms a hook is attached below one of the scale-pans; this scale-pan is often attached to the beam by shorter chains than the other and the body to be weighed can be suspended from the hook.

In other forms, as shewn in Fig. 53 (*a*) (p. 98), a wooden stand or bridge rests on the floor of the balance case over one of the scale-pans which can swing freely below it, the supports of the scale-pan passing on either side of the bridge; the body to be weighed is suspended from a hook attached to the knife edge which carries the scale-pan. The vessel of liquid is placed on the bridge and the body can be immersed in it.

A spring balance often forms a convenient instrument for use as a hydrostatic balance. Jolly's balance is a special form of spring balance so used.

In many of the experiments to be described the solid has to be weighed in water. This introduces several sources of error.

If the solid is very light the weight of the wire or thread by which it is supported may need to be considered, moreover a capillary force is exerted on the wire where it cuts the liquid; it is therefore important that the supporting wire should be as fine as is consistent with strength. A horsehair or a piece of fine wire may be used, a piece of thread will serve but it absorbs moisture and so varies in weight as it gets wet.

Water again contains dissolved air, this collects in bubbles on the sides of the containing vessel or of any body immersed in the water and the apparent weight of the body is reduced by the buoyancy of these bubbles. The bubbles must be carefully brushed off the solid before weighing; for very accurate work it is desirable to boil the water and allow it to cool previous to use, or it may, if convenient, be placed for a time under the exhausted receiver of an air-pump. These precautions are clearly specially necessary in the case of a small body.

We proceed now to consider some experiments with the Hydrostatic balance.

EXPERIMENT 16. *To find, by the Hydrostatic balance, the specific gravity of a solid body which sinks in water, and to determine its volume.*

Suspend the body by a piece of fine wire from the pan of the balance and weigh[1] it. Let the weight be W grammes.

[1] In all these experiments several observations of the weight must be taken. For exact work the "method of oscillations," Glazebrook and Shaw, *Practical Physics*, § 12, p. 107, should be employed. For precautions to be observed in determining specific gravities, see Glazebrook and Shaw, *Practical Physics*, §§ 15–19.

Immerse the body in water and weigh it again. Let the weight be W' grammes; then $W-W'$ measures the upward thrust of the water on the body, and this, we know, is equal to the weight of water displaced. This water is clearly equal in volume to the body. Thus W grammes is the weight of the body and $W-W'$ grammes is the weight of an equal volume of water.

Now Specific gravity

$$=\frac{\text{weight of body}}{\text{weight of equal volume of water}}=\frac{W}{W-W'}.$$

The volume of the body which is equal to that of the water displaced[1] is $W-W'$ cubic centimetres.

Thus, in the case of a piece of glass, the following observations were made:

Weight in air	$=76{\cdot}8$ grammes.
Weight in water	$=46\ 32$ grammes.
Weight of water displaced	$=30{\cdot}48$ grammes.

$$\text{Specific gravity}=\frac{76{\cdot}8}{30{\cdot}48}=2{\cdot}52.$$

Also volume of glass $=30{\cdot}48$ cubic centimetres.

EXPERIMENT 17. *To find, by the Hydrostatic balance, the specific gravity of a liquid.*

Weigh a body, say a piece of glass, in air; let the weight be W grammes. Then weigh it in water. Let the weight be W_1 grammes; weigh it in the liquid whose specific gravity is required, let the weight be W_2 grammes. Then $W-W_2=$ weight of a quantity of liquid equal in volume to the body, and $W-W_1=$ weight of a quantity of water equal in volume to the body.

Hence the specific gravity of the liquid $=\dfrac{W-W_2}{W-W_1}$.

[1] In this statement the variation in the density of water with temperature is neglected.

Thus, using the same piece of glass as in the last experiment:

Weight in air	$= 76{\cdot}8$ grammes.
Weight in water	$= 46{\cdot}32$ grammes.
Weight in liquid	$= 43{\cdot}88$ grammes.
Weight of water displaced	$= 30{\cdot}48$ grammes.
Weight of liquid displaced	$= 32{\cdot}92$ grammes.

$$\text{Specific gravity} = \frac{32{\cdot}92}{30{\cdot}48} = 1{\cdot}08.$$

EXPERIMENT 18. *To determine with the Hydrostatic balance the specific gravity of a solid body lighter than water.*

For this purpose a sinker is attached to the body of such a weight that the combination will sink in water.

(i) Weigh the solid in air, let the weight be W, weigh the sinker in air, let the weight be w, weigh the sinker in water, let the weight be w', weigh the combination in water, let the weight be W'. Then

Weight of water displaced by sinker $= w - w'$.

Weight of water displaced by combination $= W + w - W'$.

Weight of water displaced by solid

$$= W + w - W' - (w - w') = W - W' + w'.$$

Hence

$$\text{Specific gravity} = \frac{\text{weight of body}}{\text{weight of equal volume of water}}$$
$$= \frac{W}{W - W' + w'}.$$

Thus we have the following observations for a piece of wax.

Weight of solid	$= 26{\cdot}65$ grammes.
Weight of sinker (copper)	$= 11{\cdot}38$ grammes.
Weight of sinker in water	$= 10{\cdot}10$ grammes.
Weight of combination in water	$= 9{\cdot}16$ grammes.

Hence

Weight of combination in air	$= 38{\cdot}03$ grammes.

Weight of water displaced by combination = 28·87 grammes.
Weight of water displaced by sinker = 1·28 grammes.
Weight of water displaced by wax = 27·59 grammes.

$$\text{Specific gravity of wax} = \frac{26 \cdot 65}{27 \cdot 59} = \cdot 966.$$

(ii) *Otherwise thus.*

If it is convenient to support the sinker below the solid so that it can be immersed in water while the solid is not, the following method requires fewer weighings than that given above.

Weigh the solid in air, let the weight be W.

Attach the sinker below and weigh again with the sinker only in water, let the weight be W_1.

Raise the vessel containing the water so that the solid is immersed as well as the sinker and weigh the combination in water, let the weight be W'.

Then, W_1 = weight of solid + weight of sinker
− weight of water displaced by sinker.
W' = weight of solid + weight of sinker
− weight of water displaced by sinker
− weight of water displaced by solid.
$W_1 - W'$ = weight of water displaced by solid.

Hence $$\text{Specific gravity} = \frac{W}{W_1 - W'}.$$

58. The Common Hydrometer

The principle of this instrument is most easily understood by considering a hollow cylindrical glass tube loaded at one end, so that it will float vertically in water and some other fluids, and having a graduated scale of millimetres either fixed inside or engraved on the glass; the scale is adjusted so that its zero may coincide with the bottom of the tube. Observations are made by noting the depth to which the hydrometer is immersed; this depth will measure the volume of fluid displaced.

Since the weight of the hydrometer remains unchanged, the

weight of fluid displaced is the same, whatever fluid it be immersed in.

Hence, if the tube sink to depths d, d' in two different fluids, the densities of these fluids are inversely[1] as d to d'. If the first fluid be water, the specific gravity of the second is d/d'.

Suppose, for example, that the instrument is so adjusted that it sinks to division 100 in water, and to division 92 in a solution of salt in water. Then the specific gravity of the solution is 100/92 or 1·086.

Now an instrument such as this would be far from sensitive. A change of 1 mm. in the position in which it rests would mean an alteration of 1 per cent. in the specific gravity, and for accurate work this would be useless. Let us suppose however it were possible to have the tube over a metre long and let the reading in water be 1000, the reading in the salt solution 922, then the specific gravity is 1000/922 or 1·085, an alteration of a millimetre in the reading will now mean an alteration of about one in a thousand, the instrument is more sensitive, but it is too long to be of use. In such an instrument however it is only the upper part of the stem which need be graduated. Thus if the graduations extended down to 800 the specific gravities of fluids from 1 to 1·250 could be measured, the rest of the stem will never rise out of the fluid. There is no need therefore for the lower part of the instrument to take the form of a straight stem at all. It may be made in any convenient form provided that its weight and volume are the same as those of the straight stem it replaces and also that it is so constructed as to float with the stem vertical.

In practice then the hydrometer usually takes the form shewn in Fig. 61. A is a hollow glass tube ending below in a bulb C which contains mercury so adjusted as to make the instrument float in a vertical position. The upright stem B is hollow and contains a paper scale. The scale is usually not divided into equal parts as in the simple form of apparatus described above, but in such a way that the readings of the scale may give directly the specific gravity of the fluid in which the instrument is immersed.

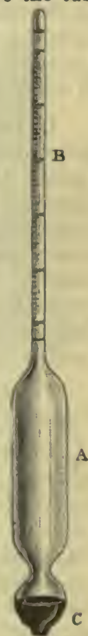


Fig. 61.

Let us suppose it is to be used for determining the specific gravities of fluids denser than water,

[1] See Section 51, Prop. 24 (Coroll.).

then the instrument rises as the fluid in which it is immersed is made more dense. The mercury in the bulb and the position of the scale then are adjusted so that, when floating in water, the top division of the scale which is marked 1·000 is level with the surface.

The scale is usually adjusted so that each division is equivalent to a change of ·001 in the specific gravity of the fluid, and each tenth division is marked; the divisions of the scale are not equal, but decrease in length as the bottom of the stem is approached.

It is not difficult to shew how the points on the scale may be obtained by calculation; in practice, however, it is simpler to determine a few points by immersing the instrument in turn in fluids of known specific gravities; the distances between these points are then subdivided by eye.

If the instrument is to be used to determine the specific gravity of fluids lighter than water, the mercury is adjusted so that it floats in water immersed just up to the bottom of the stem; when immersed in fluids of less density it sinks further; thus the graduation 1·000 is at the bottom, and the graduations run up the tube, becoming less towards the top.

Fig. 62 (i), (ii), shew an hydrometer floating in water and in spirit.

(i) Fig. 62. (ii)

In order to get sufficient range without having very long stems, a series of hydrometers is generally employed. These are adjusted in such a way that one instrument may sink just up to the top of the stem in a given fluid, while the next will float in the same fluid with the whole of the stem exposed. Thus the scale runs on from one instrument to the next, and by a

proper choice of the hydrometer the specific gravity of any given fluid can be determined.

EXPERIMENT 19. *To use the common hydrometer to find the specific gravity of a liquid and to check its graduations.*

Immerse the hydrometer in the liquid and note the reading. This gives the specific gravity. To check the graduations determine in some other way the specific gravity of the liquid, e.g. by the hydrostatic balance, Section 57, or by the specific gravity bottle; the two results should agree.

59. Nicholson's Hydrometer.

This instrument, shewn in Fig. 63, consists of a hollow bulb to which a thin stem is attached, the stem carries a tray or cup, into which weights can be placed. Below the bulb hangs a second tray or basket. This is weighted with mercury, which is adjusted so that the instrument may float vertically in water.

Fig. 63.

On the stem there is a mark, when the instrument is used it is loaded so that this mark is just in the surface of the liquid in which it is floating. Thus whatever be the liquid the volume displaced is always the same.

EXPERIMENT 20. *To weigh a body and to determine its specific gravity by the use of Nicholson's Hydrometer.*

Place the hydrometer in a tall vessel of water and take care that it floats freely and that no air-bubbles are attached to it. Weights are to be placed in the upper cup to sink the hydrometer down to the mark. To avoid the inconvenience caused by these weights falling into the water, the top of the vessel is covered with two pieces of glass[1] which fit together and close it. The stem of the instrument rises through a hole which has been drilled in the glass.

Place weights on the upper cup till the instrument sinks to

[1] The glass is not shewn in the figure.

the mark. It will be found that there is not a perfectly definite position of floatation for given weights, the hydrometer will, within limits, rest in any position. This is due chiefly to the capillary action between the water and the stem, the position will be more definite if the stem is free from grease; it may be cleaned before the instrument is used by being rubbed with cotton-wool soaked in alcohol.

Suppose the instrument is floating with the mark just below the surface. Take off some small weights till the mark just rises above the surface and note the weight left on; put on weights until the mark again sinks below and note the weights; do this several times and take the mean. Let it be W_1.

Now place the solid in the upper cup; the instrument sinks. Take off weights until the mark is again in the surface. Determine as above the exact weight to be taken off. Let it be W, then W is clearly the weight of the solid.

Place the solid in the lower pan, taking care that no air-bubbles adhere, the weight supported is the same as before, but the solid is now acted on by the buoyancy of the liquid displaced, the instrument therefore rises. Place additional weights on the upper pan until it again sinks to the mark, determining their value as before. Let it be W', then W' is clearly the weight of liquid displaced by the solid.

Hence the specific gravity is W/W'.

Instead of reckoning the weights taken off in the second operation and those added in the third, it may be more convenient to reckon the weights on in each case.

Let them be W_1 when the solid is not in either pan, w when the solid is in the upper cup, w' when it is in the lower cup.

Then the weight of the solid is $W_1 - w$ and the weight of water displaced $w' - w$.

Hence the specific gravity is $(W_1 - w)/(w' - w)$.

Thus in an experiment with sulphur the hydrometer was in adjustment with 8·35 grammes in the cup, the weights taken off when a piece of sulphur was placed in the upper pan were 5·81 grammes. This then was the weight of the sulphur; the weights added to these when the sulphur was transferred to

the lower pan were 2·92 grammes, giving the buoyancy of the sulphur.

Thus specific gravity of sulphur $= 5{\cdot}81/2{\cdot}92 = 1{\cdot}99$.

EXPERIMENT 21. *To find the specific gravity of a liquid with Nicholson's hydrometer.*

For this purpose we require to know the weight of the hydrometer; let it be W_0.

Place the instrument in water and determine as before the weight required to sink it up to the mark. Let it be W_1. Place the instrument in the liquid and let the weight required to sink it be W_2. The weight of water displaced, since the hydrometer is floating, is $W_0 + W_1$. The weight of liquid displaced is $W_0 + W_2$. The volumes of these two weights are the same, each being equal to the volume of the hydrometer up to the mark.

Thus the specific gravity of the liquid is

$$(W_0 + W_2)/(W_0 + W_1).$$

In an experiment with a solution of salt in water containing 20 grammes of salt in 100 grammes of the solution, the weight of the hydrometer was 11·27 grammes, the weight required to sink it in water was 8·35 grammes. Thus the weight of water displaced was 19·62 grammes. The weight required to sink it in the salt solution was 9·88 grammes, thus the weight of salt solution displaced was 21·15 grammes.

Hence specific gravity of salt solution

$$= 21{\cdot}15/19{\cdot}62 = 1{\cdot}078.$$

60. Jolly's Balance.

This, as shewn in Fig. 64, consists of a long spiral spring which carries two light scale-pans one below the other. A vertical scale graduated on mirror glass is mounted behind the spring and a white

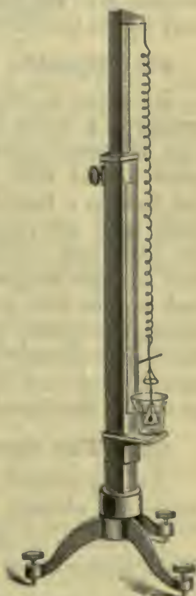


Fig. 64.

bead is attached to the end of the spring. The division of the scale opposite to the top of the bead can be read accurately, by looking at it with one eye closed, from such a position that the bead exactly covers its own image in the mirror. When using the apparatus the lower pan is kept always immersed in a vessel of water. The method of making measurements and the theory of the instrument are much the same as with Nicholson's hydrometer. Weights are placed in the upper pan until the bead comes opposite to some convenient division of the scale. The body to be weighed is then placed in the upper pan and weights removed until the bead is in the same position as before; this gives the weight of the body. The body is then transferred to the lower pan, thus causing the bead to rise, weights are added until the bead again occupies its sighted position; these weights give the buoyancy, and by dividing the weight of the body by its buoyancy we get the specific gravity of the body[1].

61. Specific Gravity Balls.

In order to obtain a rapid determination of the specific gravity of a liquid of which only a small quantity is available, a set of specific gravity balls is useful. These are small glass bulbs loaded with mercury, and so adjusted that each will just float in a liquid of definite specific gravity. A number of these balls are placed in the liquid to be examined, some of the balls sink, others float; if one be found which will just float the specific gravity of the liquid is known. If it happens that no one ball just exactly floats, limits can be found by noting the specific gravity of that ball, among those that sink, which most nearly floats; and the specific gravity of the ball, among those that float, which most nearly sinks. The specific gravity of the liquid lies between these two.

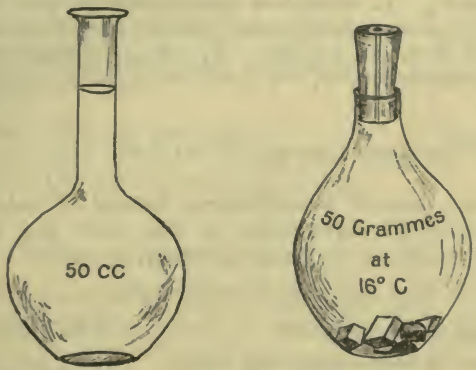
62. The Specific Gravity Bottle.

An experiment involving the use of the specific gravity bottle has already been described (see Section 9).

Two forms of bottle are shewn in Fig. 65 (i) and (ii). In (i) the bottle has a narrow neck which is open; a fine mark is

[1] For practical details see Glazebrook and Shaw, *Practical Physics*, p. 137.

made on the neck and in use the bottle is always filled exactly up to this mark. In (ii) the bottle is closed with a ground



(i) Fig. 65. (ii)

glass stopper, which is perforated with a small hole. The bottle is filled to the top of the neck with water or whatever other liquid is being used, and the stopper is then pushed home. The surplus liquid escapes by the hole in the stopper, leaving the bottle completely filled.

The specific gravity bottle is used to find the specific gravity of a liquid or of a solid which is either in the form of a powder or can be broken into small fragments and inserted in the bottle.

Bubbles of air easily collect among the fragments or on the sides of the bottle; care must therefore be taken to free the liquid used from air as far as possible.

The following experiments may be performed with the specific gravity bottle.

EXPERIMENT 22. *To find the specific gravity of a liquid with the specific gravity bottle.*

Weigh the bottle empty[1]. Let its weight be W grammes;

[1] Before the bottle is weighed empty it should be dried. This is done by connecting a glass tube with a bellows and blowing air into the bottle. The tube is held over a Bunsen burner to heat the air before it enters the bottle.

then fill it with water and again weigh it. Let the weight be W_1; finally fill it with the liquid and weigh it. Let the weight be W_2. In these last operations care must be taken that all traces of air-bubbles are removed from the sides of the bottle and from the stopper. $W_2 - W$ is the weight of liquid which fills the bottle, $W_1 - W$ is the weight of an equal volume of water.

Hence the specific gravity of the liquid $= \dfrac{W_2 - W}{W_1 - W}$.

In some cases it is convenient, instead of weighing the empty bottle, to counterpoise it with shot and lead foil. Then two weighings only are necessary.

The following observations were made.

Weight of empty bottle	6·85 grammes.
Weight of bottle filled with water	31·82 grammes.
Weight of bottle filled with liquid	30·43 grammes.
Hence	
Weight of water filling bottle	24·97 grammes.
Weight of liquid filling bottle	23·58 grammes.

Specific gravity of liquid $= \dfrac{23 \cdot 58}{24 \cdot 97} = \cdot 944$.

EXPERIMENT 23. *To find the specific gravity of a solid in small fragments with the specific gravity bottle.*

Weigh the fragments of the solid; let the weight be W. Fill the bottle with water and place it with the solid on the pan of the balance. Let the combined weight be W_1. Place the solid in the bottle and fill it up with water, taking care to get rid of all air-bubbles; let the weight be W_2. This weight W_2 will be less than W_1 because water will have been displaced from the bottle by the solid and $W_1 - W_2$ will be the weight of the water displaced. The volume of this water is equal to that of the solid.

Hence specific gravity of solid $= \dfrac{W}{W_1 - W_2}$.

The following observations were made.

Weight of solid	5·67 grammes.
Weight of solid, bottle and water	37·49 grammes.
Weight of solid in bottle, bottle and water	34·62 grammes.
Hence	
Weight of water displaced	2·87 grammes.

Specific gravity of solid $= \dfrac{5{\cdot}67}{2{\cdot}87} = 1{\cdot}97.$

63. Solids soluble in water.

In the descriptions just given of the various methods of determining specific gravity it has been assumed that the solids used could be immersed in water. This of course is not always the case; salt, sugar, and many other substances when placed in water dissolve. If the substance whose specific gravity is required is soluble in water some other liquid must be employed in which it will not dissolve; thus sugar may be weighed in alcohol.

The result of the observations give us the ratio of the weight of a given volume of sugar to that of an equal volume of alcohol. Let us call this ${}_s\sigma_a$. Now the specific gravity of alcohol is the ratio of the weight of a given volume of alcohol to the weight of the same volume of water, let this be ${}_a\sigma_w$.

Then we have ${}_s\sigma_a \times {}_a\sigma_w$

$$= \frac{\text{weight of 1 c.cm. sugar}}{\text{weight of 1 c.cm. alcohol}} \times \frac{\text{weight of 1 c.cm. alcohol}}{\text{weight of 1 c.cm. water}}$$

$$= \frac{\text{weight of 1 c.cm. sugar}}{\text{weight of 1 c.cm. water}}$$

$$= \text{specific gravity of sugar.}$$

Hence the specific gravity of sugar is found by multiplying its specific gravity relative to alcohol by the specific gravity of alcohol.

Another method of procedure is to weigh the body, then to

coat it completely with a small quantity of wax of known specific gravity and then to find the specific gravity of the compound body. From this, since the weight of the body, the weight of wax and the specific gravity of the wax are known, the specific gravity of the body can be found (see Section 11).

Examples. **(1)** *A crystal of copper sulphate weighs* 9 *grammes in air and* 5·68 *grammes in turpentine of specific gravity* ·88. *Find its specific gravity.*

The weight of turpentine displaced is 3·32 grammes.

Hence specific gravity relative to turpentine $= \frac{9}{3 \cdot 32} = 2 \cdot 71$.

Therefore specific gravity required $= 2 \cdot 71 \times \cdot 88 = 2 \cdot 38$.

(2) *A piece of sugar weighing* 32 *grammes is coated with* 3·6 *grammes of wax of specific gravity* ·9. *The weight of the whole in water is* 11·6 *grammes. Find the specific gravity of the sugar.*

The weight of the sugar and wax is 35·6 grammes.

The weight of water displaced is 24 grammes.

Therefore the volume of the whole is 24 c.cm.

The volume of the wax is 3·6/·9 or 4 c.cm.

Thus the volume of the sugar is 20 c.cm.

The mass of sugar is 32 grammes.

Hence the specific gravity of the sugar is 32/20 or 1·6.

(3) *In Archimedes' experiment, Hiero's crown, together with lumps of gold and silver equal in weight to the crown, were each weighed separately in water. The crown lost* $\frac{1}{14}$ *of its weight, the gold* $\frac{4}{77}$, *and the silver* $\frac{2}{21}$. *In what proportion were gold and silver mixed in the crown?*

From the experiment it follows that the mean specific gravity of the crown was 14, that of the gold $\frac{77}{4}$ and of the silver $\frac{21}{2}$.

Suppose that in each cubic centimetre of the crown there is v c.cm. of gold, there will be $1 - v$ c.cm. of silver. The weight of gold will be proportional to $77v/4$ and of silver to $21(1-v)/2$; the weight of a c.cm. of the whole is proportional to 14.

Hence
$$14 = \frac{77v}{4} + \frac{21(1-v)}{2},$$
$$8 = 11v + 6(1-v),$$
$$2 = 5v.$$

Therefore $v = \frac{2}{5}$ of a cubic centimetre.

Hence the crown was composed of $\frac{2}{5}$ by volume of gold and $\frac{3}{5}$ of silver.

64. The U-Tube Method.

When two liquids which do not mix are poured into the two vertical limbs of a U-tube, the heights of the columns of liquids in the two tubes above the common surface are, as we have seen, different. The pressures at the common surface are the same in the two liquids; these pressures are measured respectively by the heights of the two columns, and since the densities of the two are different the heights are different.

A method of comparing the densities of two liquids which do not mix is based on this.

The apparatus employed is shewn in Fig. 66. The U-tube is mounted on a stand, and scales of millimetres are fixed beside each limb of the tube. The heights of the columns can be read on these scales.

Experiment 24. *To compare the densities of two liquids by observations on the heights of balancing columns.*

(i) Let $ABCD$, Fig. 66, be the U-tube. Let the one limb AB contain oil, the other water, and let B be the common surface of the two. Draw BC horizontal to meet the water in the other limb in C.

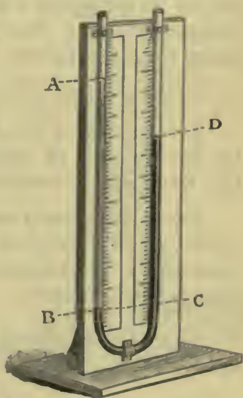


Fig. 66.

Let $AB = h$, $CD = h'$, let ω be the weight of a unit of volume of the oil, ω' of that of the water.

Then since B and C are points in the same fluid—the water—in the same horizontal line, the pressure at B is equal to that at C.

Let π be the atmospheric pressure.

Pressure at $B = \pi + \omega h$.

Pressure at $C = \pi + \omega' h'$.

Hence $$\omega h = \omega' h',$$

or $$\frac{\omega}{\omega'} = \frac{h'}{h}.$$

Thus the weights of unit volume of the two fluids, and therefore their densities, are inversely proportional to the heights of the respective columns.

If the fluid in DC be water, as we have supposed, then the ratio ω/ω' measures the specific gravity of the fluid in AB.

Hence in this case

$$\text{Specific gravity of fluid in } AB = h'/h.$$

*(ii) If the liquids mix the apparatus requires modification; one form which is then useful is shewn in Fig. 67. Two U-tubes ABC, DEF are used, one limb of each being much longer than the other. The two shorter limbs are connected together by a piece of india-rubber tubing as shewn at G. A small quantity of liquid is poured into each tube, thus enclosing some air in the part CGD. Additional quantities of liquid are then poured into the tubes in turn; the air in the space CGD is gradually compressed, and finally the columns stand as in the figure.

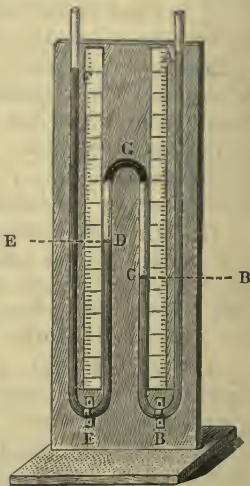


Fig. 67.

Draw CB and DE horizontal.

Let $AB = h$, $FE = h'$, and let ω, ω' be the weights of unit of volume of the two liquids respectively.

Scales are placed alongside the two tubes, and the heights h and h' can be read off on these scales. Let π be the atmospheric pressure.

Then the pressure of the enclosed air at C and D is the same.

But pressure at C = pressure at B

$$= \pi + \omega h,$$

and pressure at D = pressure at E

$$= \pi + \omega' h'.$$

Hence $$\omega h = \omega' h',$$

and as in (i) specific gravity of fluid in AB

$$= \omega/\omega' = h'/h.$$

Another form of the apparatus suitable for two fluids is shewn in Fig. 68.

EXPERIMENT 25. *To compare the densities of two liquids which will mix by means of the inverted* U*-tube.*

An inverted U-tube $ABCD$, Fig. 68, dips into two beakers containing the liquids. At the top of the tube there is an opening to which a short length of india-rubber tubing is attached. This can be closed by a clip or by the insertion of a piece of glass rod. Air is sucked out of the tube through this upper opening, and the liquids rise in the two limbs of the tube. The rise of the liquid is caused by the atmospheric pressure acting on the surface of the liquids in the beakers (see Section 68). The heights of the columns of liquid in the two tubes will be found to be different. Scales are fixed to the apparatus and by their means the heights of the columns can be measured. Measure the two heights AB, CD, reckoning from the level of the liquid in the beakers in each case, let them be h and h', and let ω and ω' be the weights of unit volume of the two liquids. Let π be the atmospheric pressure. Then the pressure of the enclosed air at B and C is the same, and the pressures at A and D of the liquids within the tubes are equal to the pressures at the same level of the liquids in the respective beakers. Each is therefore equal to π.

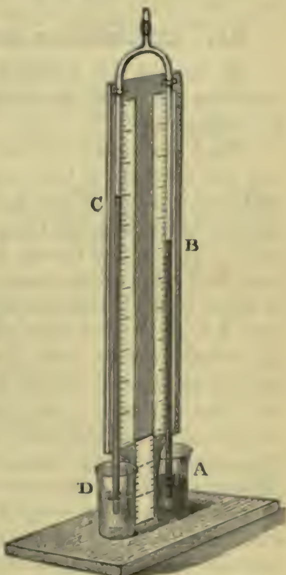


Fig. 68.

Hence from the column AB

$$\pi = \text{pressure at } B + \omega h,$$

and from column CD

$$\pi = \text{pressure at } C + \omega' h'.$$

Therefore $\qquad \omega h = \omega' h'.$

Thus, as before, the ratio of the weights of unit volume of the two liquids is inversely proportional to the ratio of the two heights. Also if the liquid in CD be water, we have

$$\text{Specific gravity of the liquid in } AB = \frac{\omega}{\omega'} = \frac{h'}{h}.$$

65. Columns in tubes of unequal Cross Section.

It should be noticed in connexion with the foregoing experiments that the area of the cross section of the tube is immaterial[1].

Each limb need not be of the same cross section throughout, and the sections of the two may be quite different; the results will be the same. Thus, to take the simple case of Fig. 66, the pressure at B, that is, the thrust per unit area over a horizontal section at B, is equal to the thrust per unit area over a horizontal section at C, but the whole thrust over the cross section at B need not be equal to the whole thrust over the cross section at C, for the areas of these two cross sections may be very different. It does not follow therefore that the weight of the fluid AB is equal to that of the column CD. The true statement is that the weight of a column of height AB and unit cross section is equal to the weight of a column of height CD and unit cross section.

Fig. 69.

[1] If the tube be very narrow capillary action will produce an effect, but apart from this the statement is true.

In dealing with the problem when the two tubes have, as in Fig. 69, very different cross sections, let us consider an inner tube described within the wider tube and of the same cross section as the narrow tube. We may suppose that the column of fluid within this tube is balanced by the thrust due to the fluid in the narrow tube; the weights of these two columns measured from the common surface, are equal. The rest of the fluid in the wide tube is supported by the vertical component of the thrust on the sides of this tube.

Example. *The cross sections of the two limbs of a* U*-tube are* 1 *square inch and* ·1 *square inch in area respectively. The lower part of both tubes contains mercury (specific gravity* 13·6). *What volume of water must be poured into the wider tube to raise the surface of the mercury in the narrow tube* 1 *inch?*

Since the cross sections of the two tubes are as 10 to 1, if the mercury rises 1 inch in the narrow tube it sinks ·1 inch in the other. Thus the upper surface of the mercury is 1·1 inches above its surface of junction with the water in the wide tube.

Hence the height of the water column is $1·1 \times 13·6$ or 14·96 inches above this surface. Hence since the cross section of the water column is 1 square inch 14·96 cubic inches of water have been poured into the tube.

EXAMPLES.

SPECIFIC GRAVITY.

[For a Table of Specific Gravities see p. 15.]

1. Find the specific gravity in the case of each of the following substances in which the first number gives the weight in air, the second the weight in water in grammes weight:

256·3,	159·1;	311·9,	195·5;
511·6,	466·6;	123·0,	116·0.

2. In experiments on the specific gravity of some bodies which float in water a sinker weighing 87·2 grammes in water is used. Find the specific gravity in the case of each of the following bodies in which the first number gives the weight of the body in air, and the second the combined weight of the body and sinker in water:

20,	23·89;	50,	42·88;
63·5,	76·3;	105,	17·5.

3. A body whose weight in air is 56 grammes and in water 35 grammes has the following weights in a series of fluids:

33·5, 30·8, 29·2, 28·6.

Find the specific gravity of each of the fluids.

4. In a Nicholson's hydrometer 30 grammes are required to sink the instrument to the mark.

The weights necessary when a certain solid is (*a*) in the upper pan, (*b*) in the lower are respectively 2·54 grammes and 10·23 grammes; find the specific gravity and the volume of the body.

5. With a Nicholson's hydrometer for which the standard weight is 13·1 grammes the weights required with a given solid are 2·02 and 4·76 grammes; find its specific gravity.

6. The weight of a Nicholson's hydrometer is 53·6 grammes. In water 30 grammes are needed to sink it to the mark, and in a certain fluid 35·75 grammes are needed; find the specific gravity of the fluid.

7. If 35 grammes are required to sink a Nicholson's hydrometer in water and 61 grammes in a fluid of specific gravity 1·33; find the weight of the hydrometer.

8. A specific gravity bottle when filled with water is found to weigh 53·2 grammes. Some crystals weighing 2·6 grammes in air are then put in and the whole is found to weigh 54·75 grammes; find the specific gravity of the crystals.

9. An empty specific gravity bottle weighs 25·22 grammes; when filled with water the weight is 75·23 grammes, when filled with various liquids it is respectively 78·41, 71·23 and 76·85; find the specific gravities of the liquids.

10. The weight of a bottle full of water is 75·23 grammes. When crystals weighing 8·60 grammes in air are inserted the weight is 79·69; find the specific gravity of the crystals.

11. If 6·432 grammes of felspar are inserted into the same bottle the weight is 79·338 grammes; find the specific gravity of the felspar.

12. A piece of gold and a piece of silver are suspended from the two arms of a balance and are in equilibrium when the silver is immersed in alcohol, the gold in nitric acid. Compare the masses of the two.

13. The lowest graduation on the stem of an hydrometer is 1·000, the highest is 1·200; find the specific gravity of a fluid in which the instrument floats with the stem half immersed.

14. A solid whose specific gravity is 1·85 is weighed in a mixture of alcohol (specific gravity ·82) and water. It weighs 28·8 grammes in air and 14·1 grammes in the mixture; find the proportion of alcohol present.

15. The lower portion of a U-tube contains mercury; how many centimetres of alcohol (specific gravity ·82) must be poured into one limb to raise the mercury 2·5 cm. in the other?

16. When a hydrometer floats in water 1 inch of the stem is exposed, when it floats in a liquid of specific gravity 1·2, 11 inches are exposed. How much will be exposed if it be placed in a liquid of specific gravity 1·1?

17. Shew that the volume of a body can be calculated from its weight in air and its weight in water, if the density of water be known.

18. A cubic foot of water weighs 1000 ounces, and 288 cubic inches of a certain substance weighs 128 lbs.; what is the specific gravity of the substance?

19. A piece of stone weighs 3 grammes in air, and 2 in water; find its specific gravity, and its volume.

20. A certain piece of lead weighs 30 grains in water. A piece of wood weighs 120 grains in air and when fastened to the lead the two together weigh 20 grains in water. Find the specific gravity of the wood.

21. The weight of a body is 25 grammes, when weighed in water at 4° C. it weighs 20 grammes. Shew that its volume is 5 c. cm. and its specific gravity 5. Explain the statement that its density is 5.

22. A lump of copper weighing 16 ounces is placed in a tumbler full of water, and causes 1·8 ounce of water to overflow: calculate the specific gravity of copper.

23. A lump of metal weighs 10 ounces, 8 ounces in water, and 7 ounces in a certain fluid; find the specific gravity of the fluid.

24. A piece of wood weighs 12 ounces, a piece of lead weighs $5\frac{1}{2}$ ounces, the lead weighs 5 ounces in water, the lead and wood together weigh 2 ounces in water. Find the specific gravity of the wood.

25. A beaker of liquid is placed in the scale-pan of a balance and counterpoised by 253 grammes. A cube of glass each of whose edges is 25·4 mm. in length is suspended by a very fine string from a separate support so that it is immersed in the liquid, and the counterpoise has to be increased in consequence to 265·9 grammes. Find the specific gravity of the liquid.

26. A lump of metal is known to consist of silver and gold, but it is not known how much is gold and how much is silver. The metal weighs 20 grammes in air and 18·7 in water, how much gold is there in the mixture?

27. Explain the principle of the common hydrometer.

The specific gravity corresponding to the lowest mark on the stem of a certain hydrometer is 1·8. What must be that corresponding to the highest mark if the reading midway between the two is 1·6?

28. The volume of a hydrometer is 10 c.cm. and its weight 7·5 grammes. Find how much of it will be unimmersed when set to float in a liquid of specific gravity 0·880.

29. Describe Nicholson's hydrometer and shew how to use it to determine the specific gravity of a solid lighter than water.

30. The lower portion of a U-tube with vertical limbs contains mercury (specific gravity 13·6). Some liquid is poured into the right-hand limb till it occupies 12 inches of the tube. The difference of level on the two sides is found to be 10 inches. What is the specific gravity of the liquid?

31. Explain how to compare the densities of two liquids which do not mix by means of a U-tube. Mercury is placed at the bottom of such a tube and water sufficient to occupy a length of 54 cm. of the tube is poured into one limb. By how much will the level of the mercury be altered and how much oil must be poured into the other limb to bring it back to its original position?

32. The lower portion of a U-tube contains mercury. How many inches of water must be poured into one limb of the tube to raise the mercury 1 inch in the other, assuming the specific gravity of mercury to be 13·6?

CHAPTER VII.

THE PRESSURE OF THE ATMOSPHERE.

66. Density of the Air.

The fact that air has weight was first proved by Otto Guericke, the inventor of the air-pump in 1650, and may be shewn as follows. Two large spheres of glass are suspended from the pans of a balance and counterpoised. The spheres should be nearly equal, both in weight and volume[1]. One of them, Fig. 70, has a nozzle attached, by means of which it can be connected to an air-pump and exhausted. This is done and the sphere is replaced on the balance. It is found to be lighter than it was previously, its weight is reduced by the weight of the air it contained. The weight of a given volume of air depends (see *Heat*, Section 78 and also Chapter VII.) on its pressure and temperature. Under standard circumstances when the temperature is the freezing-point of water, and the pressure that due to a head of 76 centimetres of mercury, it has been shewn that a litre (1000 c.cm.) of dry air weighs

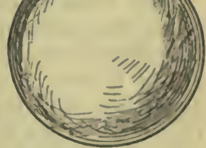


Fig. 70.

[1] By this means the correction for the buoyancy of the air is reduced. See Section 52.

1·293 grammes. Thus in these circumstances the density of air is ·001293 grammes per cubic centimetre.

67. Density of different gases.

If the one globe in the experiment just described be filled with air, while the other is filled with different gases in turn, the pressure and temperature being kept constant, it will be found that equal volumes of different gases have different masses.

This can be shewn in the following way; suspend two beakers, approximately equal in size, from the arms of a balance; let the open end of one beaker be downwards, that of the other being upwards; if the weights of the two be not exactly equal, counterpoise the heavier with shot or sand. Allow coal-gas to pass from an india-rubber tube into the first beaker. The gas fills the beaker, displacing the air, and the balance arm rises, shewing that the coal-gas is lighter than air. Then remove the coal-gas and restore the balance. Now pass carbonic acid gas into the second beaker, it sinks; the carbonic acid is heavier than the air it displaces.

68. Pressure of the Air.

Various experiments can be performed to shew that air can exert a thrust on a surface with which it is in contact. Thus

(i) Close a small bladder and place it under the receiver of an air-pump; on exhausting the receiver the bladder swells and finally bursts; or again, close one end of a glass tube with a piece of thin sheet india-rubber, connect the open end to the air-pump and exhaust; the india-rubber is forced into the tube and bursts.

Fig. 71.

(ii) Depress a beaker or tumbler, mouth downwards, into water, it will be found that the surface of the water within the beaker is below that outside.

(iii) The effect of the pressure due to the atmosphere is shewn in von Guericke's ex-

periment with the Magdeburg hemispheres, Fig. 71. A receiver is formed by two hemispheres which fit so closely together as to be airtight. When the receiver is full of air they can be easily separated. On exhausting the receiver however very great force is needed to separate the two halves. In von Guericke's original experiment it is said that a team of 16 horses was needed to pull the two hemispheres apart.

(iv) Dip a tube into water or mercury and exhaust the air in the upper part of the tube. The liquid rises in the tube in consequence of the pressure of the air on its free surface. Galileo discovered that water could not be raised in this manner more than 18 Italian ells, about 33 feet. Thus the pressure of the air is equivalent to a head of water of about 33 feet. Torricelli suggested the use of a head of mercury rather than of water to measure the atmospheric pressure, and this idea was carried out by Viviani in 1643 and is exemplified in the barometer.

69. The Barometer.

A glass tube about a metre long and 1 to 1·5 cm. in diameter, is closed at one end and filled with clean dry mercury. Care must be taken to expel from the tube all traces of air. For this purpose close the open end of the tube with the thumb, leaving a small quantity of air above the mercury. Then, by inclining the tube gently, pass this bubble of air from end to end and thus include in it the small bubbles of air which adhere to the glass; in this way nearly all the air can be got rid of[1].

Now fill the tube completely, close the open end with the thumb in such a way as to leave no air between it and the mercury; invert the tube and place the lower end below the surface of the mercury in a small trough. If the thumb be now removed the mercury will descend in the tube, but after a few oscillations, remain stationary at a height of about 76 centimetres above the mercury in the trough. If sufficient

[1] For accurate instruments the remainder of the air is removed by heating the mercury in a suitably constructed furnace till it boils in the tube.

care has been taken the space in the tube above the mercury is a vacuum, except for the presence of a little mercury vapour. On inclining the tube gently the mercury will rise to the top and completely fill it. On again placing the tube vertical it will sink to its former level.

The pressure on the upper surface at *B*, Fig. 72, is that due to the mercury vapour, and this is practically inappreciable. The pressure at *A*, a point in the tube at the same level as the mercury in the trough, is that due to the head of mercury *AB*. The pressure at a point in the free surface of the mercury, i.e. at the same level as *A*, must be the same as at *A*; this pressure is due therefore to the head of mercury *AB*. There is therefore exerted on the surface of the mercury in the trough a downward pressure measured by the height of the column *AB*. This downward pressure is due to the atmosphere, it measures the weight of a column of air of unit cross section and of height equal to that of the atmosphere.

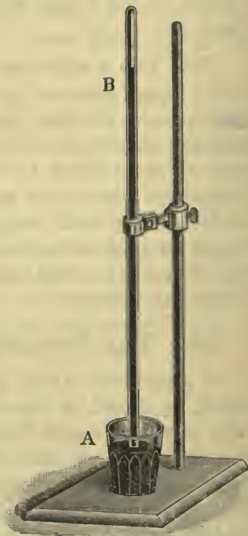


Fig. 72.

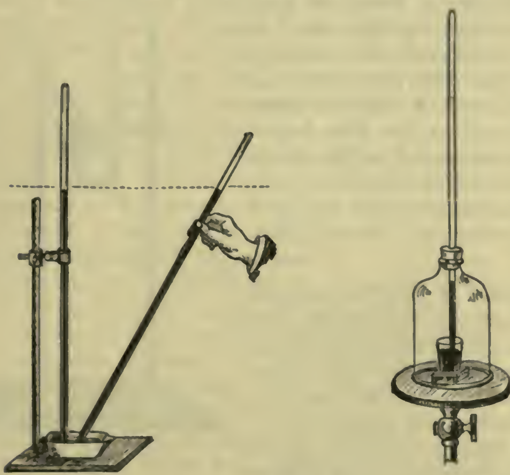
The following experiments illustrate these points.

EXPERIMENT 26. *To shew that the height of the mercury in a barometer depends on the pressure of the air.*

You are given two barometer tubes. In the one, Fig. 73 (i), the tube is moveable and can be inclined to the vertical at various angles. In the other, Fig. 73 (ii), the reservoir is under the receiver of an air-pump. Place the first with its tube vertical and note the height of the mercury in the tube above that in the reservoir. Then incline the tube at various angles to the vertical and measure the vertical height in each case. It will be

found that the vertical height of the top of the column above the mercury in the reservoir is always the same, so that though the mercury runs up the tube the level of its surface is unchanged. *The vertical height* of this column measures the pressure of the atmosphere on its base, and hence, so long as the atmospheric pressure is unaltered, this vertical height does not change.

Let us now turn to the column in the second experiment.



(i). Fig. 73. (ii).

When the receiver of the air-pump is open its height is the same as that of the barometer. Close the receiver and exhaust the air; as the air is withdrawn, and the pressure on the mercury in the reservoir thereby reduced, the level of the mercury column falls. If the process could be continued till no air were left in the receiver the levels of the mercury in the tube and the reservoir would be the same. On gradually readmitting the air to the receiver the mercury again rises in the tube till the former level is reached.

The apparatus described is shewn in Fig. 73 (i) and (ii).

70. Pascal's Experiment.

Pascal took a long glass tube of the form shewn in Figure 74; the tube is open at A and B, but can be closed at B with a close-fitting cork; each of the lengths AB, CD is greater than the barometric height. The whole is filled with mercury and placed with A downwards in a vessel of mercury. On opening the end A, while B is closed, the mercury separates into two portions as in the figure, with a vacuous space between them at B. The height of the column in AB is that of the barometer, the mercury in CD is all collected near C, and the level of the mercury in CD and CB is the same.

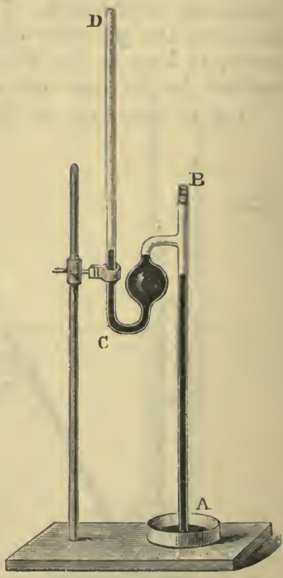


Fig. 74.

The cork is then withdrawn from B and the atmosphere thus has access to the mercury near B. The column in AB is thus driven down the tube to the reservoir, the pressure on the column in BC drives it up the tube CD, and it now stands with its upper surface at the barometric height above that in CB.

71. Fortin's Barometer.

Mercury barometers are made in various forms, according to the purposes for which they are required; the form most commonly employed in a laboratory is Fortin's, shewn in Fig. 75. In this instrument the barometer tube is enclosed in a brass tube which forms the scale for reading the height of the mercury. The cistern is attached to this scale and the instrument is suspended from a hook or other suitable support by a ring at the top. As the mercury in a barometer tube rises or falls the mercury in the cistern falls or rises.

By making the area of the tube small in comparison with that of the cistern the rise and fall in the latter can be made small; still for accurate work it is necessary either to allow for this in the graduations, or else to arrange that the zero mark of the scale may readily be brought to coincidence with the surface of the mercury in the cistern. In Fortin's barometer the bottom of the cistern is made of leather and it can be raised or lowered slightly by means of a screw shewn in the figure. The zero of the scale coincides with the point of a small ivory index which is visible above the mercury in the cistern. A reflected image of this index in the surface of the mercury can also be seen. The mercury surface is adjusted by means of the screw until the point of the index and its image appear just to touch, then the level of the mercury in the cistern coincides with the zero of the scale. The scale, as shewn in the figure, is usually only graduated from about 27 to 32 inches. A sliding vernier is attached, by means of which the scale can be read to one five-hundredth of an inch. The scale is in inches divided to twentieths, and twenty-five divisions of the vernier are equal to twenty-four of the scale. The vernier slides in a vertical slot in the upper portion of the brass tube, and through this slot the mercury is visible. At the back of the tube there is a corresponding slot in which a brass plate connected with the vernier slides. The lower edge of this plate is at the same level as the zero of the vernier; hence an observer whose eye is placed so as just to see the edge of the brass plate behind the vernier is looking in a horizontal direction. The top of the column of mercury is slightly convex and, in reading the instrument, the vernier is raised until there is a clear space above the mercury; it is then gradually lowered until the top of the mercury column, the lower edge of the plate of brass at the back, and the lower edge of the vernier all appear in the same line. By this means it is secured that the zero of the vernier is at the same height as the top of the column; for the observer, when the

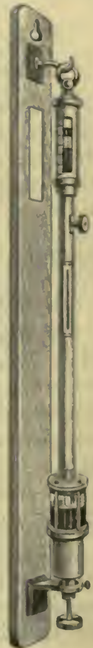


Fig. 75.

reading is taken, is necessarily looking in a horizontal direction.

*72. Corrections to the barometer reading.

The height thus read requires various corrections.

The atmospheric pressure is measured by the weight of a certain column of mercury. The weight of the column depends on its height, on the density of the mercury, and on the value of g, the acceleration due to gravity. Moreover the mercury column is depressed, though very slightly, by the pressure due to the mercury vapour above it, and by the capillary action at the sides of the tube.

i. *Correction for Temperature.*

The height of the column is measured by a brass scale. Now brass expands when its temperature rises, so that if the scale be correct at some standard temperature, such as the freezing-point of water, 0° C., it will be too long at any other temperature. But it is known that a centimetre of brass expands[1] in length by ·000019 cm. for each rise of temperature of 1°, and that the increase of length is proportional to the rise of temperature.

If we denote this coefficient by α, then, for $t°$, the increase in length of each centimetre is αt centimetres, hence if h centimetres is the height of the column as read on the scale, the true height is H where $H = h(1 + \alpha t)$.

We have thus found the true height of the column of mercury, but we need the weight of the column. Now the density of mercury decreases as the temperature rises and the decrease, for each degree of temperature, is ·000181 of the density at 0° C. Let us denote this fraction by γ, then if ρ_0 is the density at 0, the density at $t°$ is $\rho_0/(1 + \gamma t)$.

Thus at $t°$ the weight of the column is less than that of a column of equal height at 0° in the ratio 1 to $1 + \gamma t$.

Thus the height of a column at 0° which would give the same pressure as that observed is $H/(1 + \gamma t)$.

[1] Glazebrook, *Heat*, pp. 55, 62, 92.

Hence, substituting for H in terms of h we find for the height of the barometer corrected to zero Centigrade the value H_0 where,

$$H_0 = \frac{h(1+\alpha t)}{1+\gamma t}.$$

Now α and γ are so small that quantities like $\gamma^2 t^2$ or $\alpha\gamma t^2$ may be neglected.

Hence, dividing by $1+\gamma t$, we get

$$H_0 = h(1 + \alpha t - \gamma t)$$
$$= h - h(\gamma - \alpha)t.$$

Again, $\gamma - \alpha$ is very small, being for mercury and a brass scale equal to ·000181 – ·000019 or ·000162, hence the last term is very small. At any one place the value of h does not vary very greatly; thus at sea-level it may range from 70 to 80 cm. We may therefore without serious error calculate the value of $h(\gamma - \alpha)$ as though h had its mean value 76 cm. and it becomes $76 \times \cdot000162$ or ·0123 cm. We thus arrive at the following approximate rule.

To reduce the barometer reading to zero Centigrade subtract from the observed reading at t° $\cdot0123 \times t$ centimetres.

Thus, if the observed reading is 74 centimetres and the temperature 15°, the true reading is 74 – ·184 or 73·816 centimetres.

This correction is not quite accurate, it ought to be $74 \times \cdot000162 \times 15$ or ·179 cm., but the difference of five-hundredths of a millimetre is, for most purposes, inappreciable.

ii. *Correction for Capillarity.*

The capillary action always depresses the mercury column, the correction required depends on the bore of the tube and also on the methods employed in cleaning it. If the mercury has been boiled in the tube, the correction for a tube of about 1 cm. in diameter is about ·02 millimetres; this must be added to the observed height.

iii. *Correction for Vapour Pressure.*

Again, the vapour pressure of the mercury depresses the column; the correction therefore is to be added to the column,

but it is very small, being approximately equal to $\cdot 002 \times t$ mm. where t is the temperature.

iv. *Correction for Value of Gravity.*

The value of g depends on the latitude and on the height of the place of observation; it is usually referred to sea-level in 45° of latitude. It is known from the theory of the figure of the Earth that, if g be the value at a height of l metres in latitude ϕ, g_0 the value at sea-level in latitude 45°, then

$$g = g_0 (1 - \cdot 0026 \cos 2\phi - 0000002l).$$

In order to correct then to sea-level and 45° of latitude, the observed height must be multiplied by

$$1 - \cdot 0026 \cos 2\phi - \cdot 0000002l.$$

v. *Correction for Capacity of the Cistern.*

If the level of the mercury in the cistern be not adjustable, the correction for the rise and fall in the cistern may be considerable. Let us suppose the mercury in the tube rises a distance x from its standard position; the mercury in the cistern falls a distance X say. The true increase in the height of the column therefore is $x + X$. Now let a be the area of the mercury in the tube, A that of the mercury in the cistern. Since the volume ax of mercury which has entered the tube has come from the cistern it must be equal to the volume AX which has left the cistern.

Thus $$AX = ax$$

and $$X = \frac{a}{A} \cdot x.$$

Hence the true rise is

$$x + \frac{a}{A} \cdot x \text{ or } x\left(1 + \frac{a}{A}\right).$$

Thus, if a rise of x centimetres be observed, it has to be corrected, to obtain the true rise, by multiplying it by $1 + a/A$.

In some instruments this correction is avoided by making the divisions too small in the ratio 1 to $1 + a/A$.

73. Forms of Barometer.

There are various other forms of mercurial barometer. In the wheel barometer, which is very commonly used, the tube is U-shaped; the two limbs being of very unequal length. The longer limb is closed and above the mercury there is a Torricellian vacuum, the shorter limb is open and the atmospheric pressure acts on the mercury it contains. A small piece of iron or glass floats on the surface of this mercury and is partly supported by a light thread which passes over a pulley and carries a counterpoise. To the axis of the pulley is fixed a pointer which moves over a dial. Changes in the level of the mercury in the tube are thus indicated by the motion of the pointer.

The siphon barometer is an instrument similar to the above but without the weight and pointer. The bore of the two tubes is usually the same, so that the mercury falls as much in one limb as it rises in the other. Scales are however generally provided for each limb. The divisions in the upper scale to be reckoned upwards, those in the lower scale downwards.

*74. Standard Barometer.

It is never easy to read accurately the position of a mercury surface which cannot be reached, and therefore the exact determination of the height of the barometer is not very easy. In some standard instruments a small index of glass or metal is fixed in the glass tube above the surface of the mercury; the point of the index is directed downwards. The cistern below is adjustable, and the level of the mercury column in the tube can be raised or lowered until it comes in contact with the tip of the index. When this is the case the index and its image, formed by reflexion in the mercury, just coincide.

The exact position, relative to the scale, of this index can be determined once for all, and hence the level of the top of the column can be found. A similar index is fitted to the lower part of the scale and can be made to slide up and down by a rack and pinion or in some other way. To this index the vernier is attached.

The height is measured by adjusting the cistern until the upper index is in contact with the mercury in the tube, the lower index is then brought into contact with the surface of the mercury in the cistern and the scale and vernier are read. The distance between the indices is thus found with accuracy and is the barometric height.

75. The Aneroid Barometer.

The principle of this instrument is the same as that of Bourdon's gauge. (Section 39.)

A small chamber is closed with a diaphragm of thin corrugated metal and partially exhausted. Variations in the external pressure cause this diaphragm to yield to an amount proportional to the change of pressure; the motion of the diaphragm is magnified by means of a lever and transmitted to an index, by this means the variations of pressure are indicated.

In other instruments the chamber takes much the same shape as in the gauge, it is, however, closed and exhausted and the variations in its form are due to changes in the external pressure.

An aneroid barometer can be arranged to record its indications on a piece of moving paper by means of a pencil fitted to a long lever; it then becomes a barograph.

An aneroid barometer should be graduated by direct comparison with a mercury instrument; while for many purposes, owing to its great portability, it is of more use than the standard form, still its indications, specially if it be subject to rapid changes of pressure, must not be implicitly relied on. The metal diaphragm is rarely perfectly elastic; it does not therefore always take up immediately the same position for a given pressure, and changes in its form progress for some time after the change of pressure to which they are due has taken place. Changes in temperature also may produce some alteration in the reading though good instruments are usually compensated for these.

76. Measures of Atmospheric Pressure.

The standard atmospheric pressure is measured by the weight of a column of mercury at 0° C., one square centimetre in section and 76 centimetres high.

Since the weight of a cubic centimetre of mercury is 13·59

grammes weight, the atmospheric pressure per square centimetre is $13 \cdot 59 \times 76$ or $1032 \cdot 8$ grammes weight. Again, the weight of 1 gramme in London contains[1] 981 dynes or absolute C.G.S. units of force. Thus the pressure of the standard atmosphere is $1032 \cdot 8 \times 981$ or 1013177 dynes per square centimetre.

This is approximately $1 \cdot 013 \times 10^6$, or rather greater than one million dynes per square centimetre.

Now one square inch contains 6·451 square centimetres. Thus the thrust on a square inch is $1032 \cdot 8 \times 6 \cdot 451$ grammes weight. Also 1 lb. contains 453·6 grammes. Hence the pressure of the standard atmosphere in pounds weight per square inch is

$$\frac{1032 \cdot 8 \times 6 \cdot 451}{453 \cdot 6} \text{ or } 14 \cdot 69.$$

Thus there is a thrust of nearly 15 pounds weight on each square inch of the Earth's surface.

In England the standard height of the barometer is usually taken to be 30 inches.

Since 1 inch is equal to 2·54 centimetres, 30 inches is equal to 76·2 centimetres. Thus the standard height of the barometer on the metric system differs slightly from that adopted in England.

77. Water Barometer.

A barometer might be made with other liquids than mercury; water, for example, might be used, but in this case the tube would be very long; for, since mercury is 13·59 times as dense as water, the height of a water column which would balance the column of the mercury barometer would be 13·59 times its height or $13 \cdot 59 \times 76$ centimetres. This reduces to 1032·8 centimetres or 10·328 metres.

Since the weight of 1 cubic centimetre of water is 1 gramme weight, the height of the water barometer in centimetres is equal to the pressure per square centimetre in grammes weight.

[1] *Dynamics*, Section 85.

Again, 1 foot contains 30·48 centimetres, so that the height of the water barometer in feet is 1032·8/30·48.

This comes to about 33·88 feet.

Another objection to the water barometer is that the vapour pressure[1] of water is considerable and increases rapidly with the temperature. In consequence, therefore, the column will be depressed by a considerable amount and this amount will vary with the temperature; thus the correction for the pressure in the closed space above the column becomes considerable and is troublesome to apply.

The glycerine barometer is free from this disadvantage and is sometimes used.

78. Height of the Homogeneous Atmosphere.

Since the barometer column is balanced by the weight of a column of air extending from the earth's surface upwards as far as there is air, it is possible, if the density of the air be known, to calculate the height of this column. But the density of the air decreases, as we ascend, according to a complex law, and a limit to the height of the atmosphere cannot thus be found. We may, however, calculate how high the air would be if it were homogeneous throughout and of sufficient height to produce the pressure actually observed. Now this pressure is equal to the weight of 1032·8 grammes per square centimetre, and the weight of 1 cubic centimetre of dry air at freezing point and standard pressure is ·001293 grammes weight. The air column therefore must be 1032·8/·001293 or about $7{\cdot}988 \times 10^5$ centimetres.

But 10^5 centimetres is 1 kilometre.

Hence the height of the homogeneous atmosphere is 7·98 kilometres.

Again, 1 mile is 1·609 kilometres.

Thus the height of the homogeneous atmosphere is 7·98/1·609 or about 4·97 miles.

We may say then that the pressure on the Earth's surface is about the same as it would be if the earth were surrounded

[1] See Glazebrook, *Heat*, Section 116.

by an ocean of air, 5 miles in depth and of the same density throughout as the air is at the Earth's surface.

*79. Measurement of heights by the Barometer.

Observations with the air-pump have shewn (Section 69) that the reading of the barometer depends on the pressure of the air, and we have seen that air like other fluids has weight. Now the pressure at any point of a body immersed in a heavy fluid depends on the depth to which that body is immersed; if the depth be reduced by raising the body, the pressure is reduced also.

Thus consider a flexible bag tied on to the end of a glass tube as shewn in Fig. 76. Fill the bag with mercury or some other fluid and immerse it in a vessel of water, keeping the upper end of the tube above the surface of the water. The bag is squeezed by the water pressure and the mercury rises in the glass tube, becoming higher as the bag is depressed.

Fig. 76.

Or again, instead of enclosing the mercury in the flexible skin, place it in a beaker and insert in the mercury one end of a long glass tube open at both ends, then immerse the beaker and tube in a vessel of water, keeping the lower end of the tube under the mercury and the upper end above the surface of the water. The pressure of the water on the surface of the mercury drives it up the tube. A column of mercury is supported by the water pressure in just the same way as the barometer column is supported by the air. Clearly also, if the depth of the beaker be altered, there will be a relation between the alteration of depth and the change of height of the column. Alterations in depth could be measured by observing the change in height of the column.

When the liquid is water the alteration of depth is given by multiplying the change in the height of the mercury column by the specific gravity of mercury.

The same principles apply to the atmosphere. If the rise of the mercury in a barometer tube really be due to the weight

of the air above the mercury in the cistern, then when the barometer is carried up a mountain, the column will fall and there will be a relation between the amount of the fall and the height of the mountain.

This was pointed out by Pascal, and his prediction that the column would fall was verified by Clermont, who in 1648 made use of the method for the first time, to measure the height of the Puy-de-Dôme.

In employing the method to determine the difference in level of two stations the barometer is read at the two stations. From the difference in readings the weight of the column of air between the two can be calculated; hence, if the density of the air be known, the height of the column can be found, and this height is the difference of level required. It remains therefore to calculate the relation between the weight of the column and its height. Now the density of the air depends on its temperature and pressure. When the pressure is p centimetres of mercury and the temperature t° Centigrade, the density, ρ, is given in terms of ρ_0, the density at 0° C. and 76 cm., by the formula[1]

$$\rho = \rho_0 \times \frac{273}{273+t} \times \frac{p}{76},$$

but p and t both vary as the mountain is ascended and the calculation becomes complex. We may, however, assume, if the difference in level is not very great, that the column of air considered is of the same weight as it would be if the air were throughout at a uniform pressure and temperature equal to the mean of those observed at the two stations.

Observe then the pressure and temperature at each of the two stations and calculate from the above formula the value of the density corresponding to their mean, assuming the density at 0° C. and 760 mm. pressure to be ·001293 grammes per c.cm.

Let H be the required difference in level, h_1 and h_2 the two barometer readings corrected in the manner described in Section 72, then the column of air balances a mercury column of height $h_2 - h_1$. Thus if ρ be the density of the air

$$\rho H = 13{\cdot}59 \times (h_2 - h_1).$$

[1] See Sections 80, 83; also Glazebrook, *Heat*, Section 102.

Hence $$H = (h_2 - h_1) \times \frac{13 \cdot 59}{\rho}.$$

Example. *The barometer at the lower station reads* 751·9 *mm. and at the upper* 633·7 *mm., while the temperatures of the air at the two are* 13° *C. and* 7° *C. respectively. Find the difference in level.*

The difference in pressure is 118·2 mm., the mean pressure is 692·8 mm. and the mean temperature 10° C.

The mass of a column of air 1 square centimetre in section between the stations is $13 \cdot 59 \times 11 \cdot 82$ grammes. The density of air at 692·8 mm. pressure and 10° C. is

$$\frac{\cdot 001293 \times 692 \cdot 8 \times 273}{760 \times 283}.$$

Hence the difference in level is

$$\frac{13 \cdot 59 \times 11 \cdot 82 \times 760 \times 283}{\cdot 001293 \times 692 \cdot 8 \times 273} \text{ cm.},$$

or about 1413 metres.

In obtaining the result no allowance has been made for the aqueous vapour in the air. In consequence of its presence the density will be rather greater than the value used above. The difference in level, therefore, should be less.

We may obtain a more accurate formula thus:

Suppose that at the lower station the height of the barometer is h_0 and that, in going to a height of z metres, it falls to a height kh_0, k being a proper fraction.

Suppose further that the temperature is uniform and that z is so small that we may treat the density of the air as constant throughout each stratum of thickness z, though it changes as we pass from one stratum to the next.

On rising through a second distance z the barometer will fall by the same fractional amount as previously, for the fall is proportional to the average density of the stratum through which the barometer is being carried; and this average density in the second stratum bears the same relation to the pressure at its under side or kh_0 as the average density in the first stratum does to the pressure h_0. Hence, at the top of the second stratum, the height of the barometer is k^2h_0.

On rising through a third stratum the pressure falls to k^3h_0 and so on in succession.

Thus, for a series of heights in arithmetical progression, the barometer readings form a series in geometrical progression with the common ratio k.

Now, let H be the total height through which the barometer is raised, and let the distance H be divided into n-layers, each of thickness z, throughout each of which we may treat the density as constant.

Then $H=nz$, while, if h be the barometer reading at the top, $h=k^n h_0$.

Therefore
$$k^n=\frac{h}{h_0}.$$

Hence, taking logarithms,
$$n\log k=\log\left(\frac{h}{h_0}\right)=\log h-\log h_0.$$

But
$$n=\frac{H}{z}.$$

Hence
$$H=\frac{z}{\log k}(\log h-\log h_0).$$

Again, if ρ_0 be the average density of the air in the first layer of thickness z at the lower station and σ the density of mercury,

then
$$g\sigma(h_0-h_1)=g\rho_0 z,$$

for the difference in the heights of the barometer is due to the weight of a column of air of height z.

Also
$$h_1=kh_0.$$

Hence
$$h_0(1-k)=\frac{\rho_0 z}{\sigma}.$$

Moreover, if z is small, k is very nearly equal to unity, and we may write $k=1-x$ where x is very small.

Hence
$$xh_0=\frac{\rho_0 z}{\sigma}.$$

Also
$$\log k=\log(1-x)$$
$$=\log_e(1-x)\log_{10}e=-x\log_{10}e,$$

since x is very small.

Thus
$$H=\frac{z}{x\log_{10}e}\{\log h_0-\log h\}.$$

But
$$\frac{z}{x}=\frac{\sigma h_0}{\rho_0}.$$

Therefore
$$H=\frac{\sigma h_0}{\rho_0\log_{10}e}\{\log h_0-\log h\}.$$

In calculating the value of ρ_0 we take the average temperature of the column of air and assume the whole of the air in the column to be at this average temperature.

If the air at the lower station is nearly under standard conditions the value of $\sigma h_0/\rho_0\log_{10}e$ will be found to be 2×10^6 centimetres.

Hence we get the following rule for finding the difference in level between two stations.

Multiply the difference between the logarithms of the two barometric readings by two million. The result will be the difference required in centimetres.

Hence since 1 cm. = ·03281 feet, if the measurements are made in feet, the coefficient will be $\cdot 06562 \times 10^6$ or 65620 feet.

Example. *Work out by the more accurate formula the Example already solved.*

In the Example given on p. 157 we have $h_0 = 751{\cdot}9$ mm., $h = 633{\cdot}7$ mm.

The difference between the logarithms is ·07428, and the height is therefore 148560 centimetres or 1485·6 metres.

The correct result, allowing for the fact that the mean temperature is 10° C. and the pressure at the bottom 751·9 mm., obtained by using the complete coefficient $\sigma h_0/\rho_0 \log_{10} e$, is found by multiplying the above value by about 1·05; and comes to be very nearly 1560 metres. Thus the result found by assuming the air to be at standard pressure and temperature is about 74 metres or 5 per cent. too low; that given by the approximate formula on p. 157, is nearly 150 metres or 10 per cent. too low. In any case a correction is needed to allow for the aqueous vapour present. This, assuming the air to be half-saturated, would reduce the height by about 2 parts in 1000, or say 3 metres.

80. Boyle's Law.

The volumes of most bodies can be changed by change of pressure. For solids and liquids, however, this change is extremely small, and, therefore, in dealing with the dilatation of such bodies due to rise of temperature it is not necessary to notice those changes in volume which may be produced by variation of pressure.

A gas, on the other hand, alters in volume considerably for small changes of pressure, even though the temperature remain constant, and we require to investigate first the law which regulates this change. This law, called Boyle's Law, was first enunciated by the Hon. Robert Boyle in 1662.

Boyle's Law. *The pressure of a given mass of gas at constant temperature is inversely proportional to its volume.*

EXPERIMENT 27. *To verify Boyle's Law.*

In Fig. 77 AB, CD are two glass tubes connected by stout india-rubber tubing and fixed to a vertical stand.

AB is closed at its upper end and may be 50 cm. long and ·5 cm. in diameter; CD is a wider tube and is open at the top. A vertical scale parallel to the tubes is attached to the stand, and CD can slide up and down this scale. The india-rubber tubing and the lower parts of the glass tubes contain mercury. The upper part of the tube AB is filled with dry air, which constitutes the given mass of air on which the experiment is to be made. The volume of this air is proportional to the length, AB, of the tube which it occupies, and this length can be read off directly on the scale. To find the pressure of the air, let the horizontal line through B meet the mercury in the moveable tube at E, and let D be the top of this mercury column.

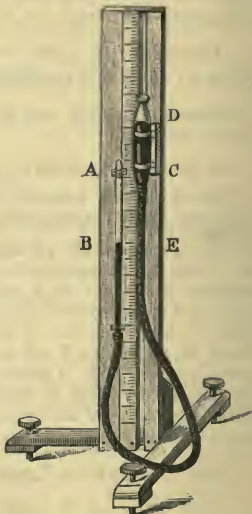


Fig. 77.

Then the pressure at B is equal to the pressure at E, and this is equal to the pressure of the atmosphere at D together with the weight of a column of mercury of unit area and height DE. Thus, if b cm. be the height of the barometer, the pressure at B is measured by a column of mercury of height $b + DE$.

Now, according to Boyle's Law, if the temperature is constant the pressure is inversely proportional to the volume. Hence if the pressure and the volume of the gas in AB be multiplied together the product obtained will be always the same however the pressure be varied.

Raise or lower the sliding tube until the mercury stands at the same level in the two tubes.

Read on the scale the level of the top of the tube AB and the position of the mercury in the tube; the difference will be proportional to the volume of the air. Since the mercury in the two tubes is at the same level the pressure of the enclosed

air is the atmospheric pressure. Observe the height of the barometer, let it be b cm.; the pressure of the enclosed air is measured by a mercury column b cm. in height. Raise the sliding tube. The mercury in the closed tube also rises but not so fast; the volume of the enclosed air is reduced, but its pressure is increased, being now measured by the height of the barometer together with the column of mercury between the two levels. Continue to raise the sliding tube until the mercury in the closed tube reaches a point B_1, half-way between A and B; let the level of the mercury in the open tube when this is the case be at D_1. The volume of the enclosed air is now half what it was. Its pressure is measured by $b + D_1B_1$. Read the levels at D_1 and B_1, it will be found that D_1B_1 is equal to b, the height of the barometer; hence the pressure of the enclosed air is now measured by $2b$; that is, it is twice what it was originally. The volume has been halved, the pressure has been doubled. Thus Boyle's Law has been verified for this case.

The verification may be made more complete by taking readings of the volume and of the pressure in a number of other positions of the sliding tube.

If B_1, D_1 be corresponding levels of the two mercury columns, the volume of air is proportional to AB_1, the pressure to $b + B_1D_1$. Set down in two parallel columns the values of AB_1, and of $b + B_1D_1$, and in a third column the numbers obtained by multiplying the corresponding values together. It will be found that these products are constant within the limits of experimental error.

The results of the experiments may be entered in a table thus:

Volume.	Pressure.	PV.
50	76	3800
40	76 + 18·8	3792
30	76 + 50·5	3795
20	76 + 113	3780

The same apparatus may be used for pressures less than that due to the atmosphere by lowering the position of the sliding tube until D is below B; in this case the pressure is given by $b - BD$.

81. Deductions from Boyle's Law.

Boyle's Law may be expressed in symbols in various ways. Thus if p be the pressure, v the volume of a given mass of gas; then, since the pressure is inversely proportional to the volume, we have the result that the ratio of p to $1/v$ is constant; denoting this constant by k we find

$$\frac{p}{\frac{1}{v}} = k.$$

Therefore $$p = \frac{k}{v},$$

or $$pv = k.$$

When we say that k is a constant we mean that it does not change when the pressure and volume are changed, if the temperature and mass of the gas are not varied. When a gas is allowed to expand under the condition that the temperature does not alter the expansion is said to be isothermal. If corresponding values of the pressure and volume are plotted the curve formed is said to be an isothermal curve.

Or again, if the volume v becomes v', and in consequence the pressure p is changed to p', since the product of the pressure and volume does not alter we have $pv = p'v'$.

Again, the volume of a given mass of gas is inversely proportional to its density; since, therefore, the volume is inversely proportional to the pressure, we see that the pressure of a gas is proportional to its density; or if ρ be the density, the ratio of p to ρ is a constant. We may write this $p = k\rho$, where k is a constant.

Examples involving Boyle's Law may be worked in various ways. Thus we may use the formula directly as in the following.

(1) *The volume of a mass of gas at* 740 *mm. pressure is* 1250 *c.cm., find its volume at* 760 *mm.*

Let v be the new volume, then from the formula $pv = p'v'$

$$v \times 760 = 1250 \times 740,$$

$$v = 1250 \times 74/76 = 1217 \cdot 1 \text{ c.cm.}$$

Or we may preferably put the argument in full thus:

Volume at pressure 740 is 1250 c.cm.

Volume at pressure 1 is 1250×740 c.cm.

Volume at pressure 760 is $\dfrac{1250 \times 740}{760}$ c.cm.

It is of course by no means necessary to measure the pressure in terms of the height of a column of mercury. Thus,

(2) *A bubble of gas* 100 *c.mm. in volume is formed at a depth of* 100 *metres in water, find its volume when it reaches the surface, the height of the barometer being* 76 *cm.*

Since the density of mercury is 13·59 grammes per c.cm., the height of the water barometer is $76 \times 13 \cdot 59$ cm., and this is very approximately 10·34 metres.

Thus the pressure at the surface is measured by a column of water 10·34 metres high, that at 100 metres by a column 110·34 metres.

Hence the new volume $= \dfrac{100 \times 110 \cdot 34}{10 \cdot 34}$ or 1067 c.mm. approximately.

(3) *The mass of a litre of air at* 760 *mm. pressure and* 0° *C. is* 1·290 *grammes. Find the mass of* 1 *cubic metre of air at a pressure of* 1·9 *mm.*

The volume of the given mass of air at a pressure of 1·9 mm. is 1000000 c.cm.

Therefore the volume at pressure of 76 cm. is

$$\frac{10000 \times 19}{76} \text{ or } 2500 \text{ c.cm.}$$

The mass of 1 c.cm. air is ·001293 grammes.

Therefore the mass of 2500 c.cm. is $2500 \times \cdot 001293$ grammes, and this is 3·24 grammes.

82. Variations from Boyle's Law.

More exact experiments have shewn that Boyle's Law is not absolutely true, though for the so-called permanent gases, oxygen, hydrogen, nitrogen and others, it holds very nearly; other gases, such as carbonic acid, which can be condensed to a liquid at ordinary temperatures by the application of a not very large pressure (see *Heat*, § 119), deviate more from the law.

83. Dilatation of Gases by Heat.

The volume of a gas changes considerably with change of temperature. In most of the problems in Hydrostatics we suppose the temperature to be constant. It will be useful, however, to state the law connecting the rise of temperature and the increase of volume, which is due to Charles and Dalton.

Law of Charles and Dalton.

The volume of a given mass of any gas increases for each rise of temperature of 1° *by a given fraction* (about 1/273) *of its volume at* 0° *C.*

Thus if the volume at 0° C. be v_0 c.cm., the increase for each rise of 1° is $v_0/273$. For a rise therefore of t° the increase in volume is $v_0t/273$, hence, if v c.cm. be the volume at t_0°, we have

$$v = v_0 + \frac{v_0 t}{273} = v_0\left(1 + \frac{t}{273}\right)$$
$$= v_0 \frac{273 + t}{273}.$$

The quantity $273 + t$ is called the absolute temperature of the gas, let us denote it by T. Then, if we denote 273 the absolute temperature of the freezing-point by T_0, the formula becomes

$$\frac{v}{T} = \frac{v_0}{T_0}.$$

Hence the volume of a given mass of gas at constant pressure is proportional to its absolute temperature[1].

It follows from the above two laws that if the pressure, absolute temperature and volume of a gas vary, then pv/T remains constant.

*84. Pressure of a Mixture of Gases.

It can be shewn by experiment that if two gases in different vessels be at the same pressure and temperature, and

[1] For an explanation of the meaning of the term absolute temperature and a description of experiments to verify Charles' Law, see Glazebrook, *Heat*, Sections 98–104.

if a communication be opened between the vessels, then the gases form a mixture in which the pressure is the same as before provided no chemical action takes place. From this result we obtain the following Proposition.

*PROPOSITION 27. *If the pressures of two gases at the same temperature t° and volume v c.cm. be p_1, p_2 respectively, the pressure of the mixture at the same temperature t° and volume v c.cm. will be $p_1 + p_2$.*

Let the volume of each gas be v.

Change the volume of the second gas until its pressure becomes p_1; its volume will be p_2v/p_1.

Let the vessels containing the two gases, each of which is at pressure p_1, communicate with each other, the pressure will remain p_1, the volume of the two gases together will be

$$v + \frac{p_2 v}{p_1} \text{ or } \frac{v(p_1 + p_2)}{p_1}.$$

Alter the volume to v, the new pressure will, by Boyle's law, become $p_1 + p_2$.

Thus the Proposition is established.

Examples. (1) *The space above the mercury in a barometer is supposed to contain some air. How would you test for this and how would you determine the correction due to the presence of the air, supposing no other barometer to be available?*

Slightly incline the tube taking care not to spill the mercury from the reservoir, the mercury rises in the tube, and if there be no air present it will when the tube is sufficiently inclined fill it completely; if air is present the mercury will not fill the tube but a bubble of air will be visible between it and the glass. In order to determine the correction without the use of another barometer it is necessary to raise the level of the mercury in the reservoir relative to the end of the tube. If it be possible to do this, Read the barometer and note the distance between the top of the column and the closed end of the tube. Let the height of the barometer be h_1. Depress the lower end of the tube in the reservoir until the distance between the closed end of the tube and the top of the column which rises in the tube is half what it was. The volume of the air above the column is thus halved, its pressure therefore is doubled. Let p_1 be the pressure due to the air before the tube was lowered, measured in mm. of mercury; after it has been lowered the pressure becomes $2p_1$. Read the height of the barometer column in the second case. Let it be h_2 and let π be the atmospheric pressure both measured in mm. of mercury.

Then in the first case, a column of height h_1 with a pressure p_1 on the top balances the atmospheric pressure; in the second case, the height of the column is h_2 and the pressure at the top is $2p_1$.

Thus $$p_1 + h_1 = \pi = 2p_1 + h_2;$$
hence $$p_1 = h_1 - h_2.$$

This gives the correction when the height of the column is h_1. Under these circumstances let the length of tube filled with air be l_1; then, when owing to a change in the atmospheric pressure the height becomes h and the length above l, the pressure due to the air is, by Boyle's Law, $p_1 l_1 / l$ or $(h_1 - h_2) l_1 / l$.

Thus the true height of the barometer will be $h + \dfrac{(h_1 - h_2) l_1}{l}$.

(2) *According to Mr Whymper's observations the barometer on the top of Chimborazo read* 14·1 *inches, while that at Guayaquil on the sea-coast below read* 30 *inches. The mean of the temperatures at the two places was nearly* 10° *C. Find the height of Chimborazo.*

The density of half-saturated air at 10° C. is given by the Tables[1] as ·001245 grammes per 1 c.cm., that of mercury is 13·59 grammes per 1 c.cm.

Hence $$H = \frac{13·59 \times 30}{·001245 \times \log_{10} e} \{\log 30 - \log 14·1\} \text{ inches}$$
$$= ·754 \times 10^6 \{\log 30 - \log 14·1\} \text{ inches}$$
$$= 20{,}600 \text{ feet.}$$

The height given by Mr Whymper as resulting from his observations after inserting all corrections is 20,545 feet.

EXAMPLES.

PRESSURE OF THE ATMOSPHERE.

[For a Table of Specific Gravities see p. 15.]

Mass of a litre of hydrogen at 0° C. and 760 mm. pressure ·0896 grammes.
Specific gravity of oxygen referred to hydrogen 16.
Specific gravity of air referred to hydrogen 14·4.

1. The volume of a mass of gas at a pressure due to 76 cm. of mercury is 500 c. cm. Determine its volume at the following pressures: 10 cm. of mercury, 5 inches of mercury, 53 inches of water, 60 fathoms of water.

2. Find the volume at 0° C. of 50 grammes of oxygen at a pressure due to 68 cm. of mercury.

3. Find the mass of 1000 cubic feet of coal-gas at standard pressure and temperature. [Specific gravity of coal-gas referred to air ·496.]

[1] Lupton's Tables (36).

4. A balloon 10 metres in diameter is filled with coal-gas at a pressure of 76 cm. of mercury. What is the weight of the balloon and its appendages if it just will not float in the air?

5. The space above the mercury in a barometer tube contains some air and the barometer reads 656 mm. when a standard barometer reads 762. Find in grammes weight per square cm. the pressure of the enclosed air.

6. There is a column of water 9 mm. in height above the mercury in a barometer, the temperature is 15° C. and the pressure of aqueous vapour at 15° C. is 12·7 mm. of mercury; find the correction on this account to the observed reading.

7. How much will a barometer rise on descending a coal-pit 50 yards deep, assuming the average temperature to be 20° C.?

8. Find the true weight of a piece of cork which weighs 75 grammes when weighed with brass weights in air.

9. Two spheres of equal volume when filled the one with air the other with hydrogen and weighed in air are found to be equal in weight, compare the pressures of the two gases.

10. Calculate the height of the mercury barometer corresponding to the following pressures: 10^6 dynes per square cm., 13·5 lbs. weight per square inch, 1500 lbs. weight per square foot.

11. Find the volumes of the following masses of air all at 0° C.:

1 gramme at 76 cm. when the pressure becomes 75·2 cm.,

300 c. cm. at 15 lbs. per square inch when the pressure becomes 76 cm.,

235 cubic inches at 60 feet of water when the pressure becomes 30 inches of mercury.

12. A barometer reads 30 inches and the space above the mercury is 5 inches. If a bubble of air which at normal pressure would occupy 1 inch of the tube is introduced what will the reading become?

13. Find the volume of 2·35 grammes of hydrogen at a pressure of 830 cm. of mercury.

14. If a vessel containing 25 grammes of hydrogen at a pressure due to 25 inches of mercury is allowed to communicate with one containing 2·5 grammes hydrogen at a pressure of 25 feet of water, find the pressure of the mixture when equilibrium is restored.

15. At what depth in water will the density of air be the same as that of water?

16. State Boyle's Law. What is the change in volume of a quantity of air which measures 20 cubic feet if the pressure change from 15 to 10 lbs. per square inch?

17. A bubble of air $\frac{1}{10}$ inch in diameter starts from the bottom of the Atlantic at a depth of two miles. Find its size on reaching the surface.

18. An air-bubble at the bottom of a pond 15 feet deep has a volume equal to $\frac{1}{1000}$ of a cubic inch, find its volume when it reaches the surface, the height of the water-barometer being 30 feet.

19. The volume of a mass of air is increased from 1 to 20 cubic feet; how is its pressure affected?

20. An elastic bag contains 25 cubic feet of air at atmospheric pressure. What will be the volume when sunk 250 feet below the surface of the sea? The height of the water barometer may be taken as 33 feet and the specific gravity of sea water as 1·026.

21. Air is contained in a cubical vessel whose edge is 4 inches long, and the pressure on a face of the vessel is 5 cwt. If air be allowed to escape freely, find the volume of the escaped air if the atmospheric pressure at that time is 15 lbs. to the square inch. [The thickness of the material may be neglected.]

22. A piston whose area is 6 square inches fits into a cylinder containing air at atmospheric pressure, namely 15 lbs. to the square inch. What force must be applied by the hand to the piston in order that the volume of the air in the cylinder may be trebled?

23. The density of air at atmospheric pressure is ·00129 grammes per c.cm. How deep must a bladder filled with air be sunk in the sea in order that its density may be equal to that of water, the specific gravity of sea water being taken as 1·025 and the height of the water barometer 33 feet?

24. A litre of air at 0° C. and under atmospheric pressure weighs 1·2 grammes; find the mass of the air required to produce at 18° C. a pressure of 3 atmospheres in a volume of 75 c. cm.

25. Explain how a measurement of the pressure of the atmosphere can be obtained from a reading of the height of the barometer.

Describe any method that you are acquainted with of obtaining an accurate barometer reading.

26. Explain the way in which the height of a mountain may be deduced from barometrical observations.

Shew that as long as the temperature of the air and the amount of aqueous vapour in it remain unaltered the ratio of the heights of the barometer at the top and bottom of the mountain will be constant.

27. Explain carefully why the mercury in a barometer falls in ascending a mountain. At sea level the barometer reads 750 mm.; on going up a mountain the reading falls to 600 mm. Compare the weights of a cubic metre of air in the two positions, the temperature being the same.

28. What experiments would you perform to shew that the height of the barometer measures the pressure of the atmosphere?

29. In a barometer with long vertical cistern the height of the mercury is 29 inches. When the tube is depressed so that the space above the mercury (which contains air) is reduced by one-half, the height of the mercury is 28 inches. What is the pressure of the atmosphere?

30. A certain bottle, which has an outlet at the bottom fitted with a tap, is nearly filled with water and is closed with an air-tight stopper. When the tap is turned on, the water runs out, at first quickly, then more and more slowly and at last not at all, though there still remains water in the bottle. Explain this and state what difference of pressure exists between the air in the bottle and that outside, when the flow has ceased.

31. What adjustments have to be made before reading the standard barometer, and what corrections have to be applied afterwards?

32. Find the height of the homogeneous atmosphere at a temperature of 15° C. when the height of the mercury-barometer is 750 mm. The specific gravity of mercury is 13·56, and that of air at 0° C. and 760 mm. ·001293.

33. Describe an experiment to prove that the pressure of the atmosphere is measured by the height of a barometer column. Find the height between two stations, having given the following data:

Density of mercury 13·6 grammes per c. cm.

Mean density of air between the two stations 0·00121 gramme per c. cm.

Height of barometer at lower station 785 mm.

" " " " upper " 630 " .

34. The height of the Torricellian vacuum in a barometer being 3 inches, the instrument indicates a pressure of 29 inches when the true pressure is 30 inches. Assuming that the faulty readings are due to the pressure of some air in the Torricellian vacuum, shew that the true reading corresponding to any faulty reading h is

$$h + \frac{3}{32 - h}.$$

35. A barometer which is known to have some air above the mercury is constructed so that the tube can be depressed into the cistern, thus varying the volume of the tube above the mercury column. When the top of the column is 6 inches below the top of the tube the barometer reads 30 inches. On depressing the tube 3·5 inches the barometer reading is reduced to 29·5 inches. Find what the reading would be if there were no air above the mercury.

36. A bent uniform tube has two equal vertical branches close together, one end being open and the other closed. Mercury is poured into the open end, no air escaping. If when the mercury just fills the open tube the air occupies two-thirds of the closed branch, prove that the length of either branch is equal to three times the height of the mercurial barometer.

CHAPTER VIII.

HYDROSTATIC MACHINES.

85. The Pipette.

This instrument is shewn in fig. 78. It consists of a glass tube with a bulb blown on to it about one-half of the way down; the lower end of the tube is drawn out to a point and ends in a small orifice. It is used for removing liquid from one vessel to another. The lower end is placed in the liquid, which is drawn up into the pipette by suction at the top. When there is a sufficient quantity in the bulb the mouth is removed and the upper end closed with the finger. The lower end is then withdrawn from the vessel and the atmospheric pressure acting at the orifice is sufficient to retain within the bulb the liquid it contains. The lower end is then placed in the vessel to which the liquid is to be transferred and the finger removed from the top; the liquid then flows out, and its flow may be hastened by blowing into the upper end.

In some cases pipettes are constructed to hold measured quantities, 10, 25 or 50 c.cm. of liquid. A mark is made on the upper part of the tube, and the volume, when the instrument is filled up to the mark, is indicated on the bulb.

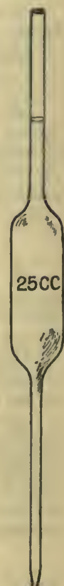


Fig. 78.

86. The Siphon.

This is an instrument by means of which we can empty a vessel filled with liquid without moving the vessel.

It consists of a bent tube ABC, Fig. 79, open at both ends, one limb being longer than the other. It is filled with water or whatever liquid the vessel to be emptied contains; the ends are then temporarily closed with the fingers and the shorter limb AB is placed below the surface of the liquid in the vessel it is desired to empty.

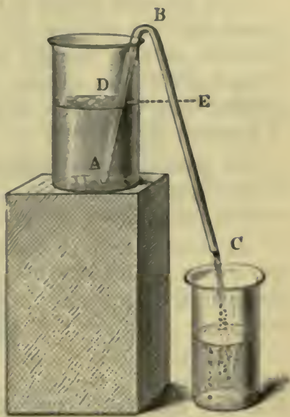


Fig. 79.

The other end C is outside the vessel and below the level of the liquid surface. On opening the ends at A and C, the liquid flows through the siphon from A to C, as shewn in the figure.

To explain the action of the siphon, let us suppose it to be filled with liquid and the end C closed. Let it cut the surface of the water in the vessel in D and let DE be horizontal. Let us consider the forces acting on the column of liquid EC.

Since D and E are points in the same horizontal plane in a liquid at rest, the pressure at E is equal to the pressure at D.

But the pressure at D is the atmospheric pressure; hence the pressure at E is the atmospheric pressure acting downwards on the column EC.

Thus the downward forces on EC are the thrust at E due to the atmospheric pressure acting on the area of the top of the column, and the weight of the column; these forces are balanced by the upward thrust exerted by the finger at C which closes the end of the tube, this upward thrust then must be greater than that due to the atmospheric pressure acting on the end of the column EC.

If the end C be opened the pressure at C becomes equal to the atmospheric pressure, thus at the moment of opening the column EC is acted on downwards by the thrust due to the

atmospheric pressure at E and its weight, and upwards by the thrust due to the atmospheric pressure at C.

This upward thrust is too small to balance the downward thrust and the column EC begins to move downwards. The pressure at E is thereby lessened, and if we now consider the column DBE, the upward thrust at D on this column becomes greater than that at E and the liquid is forced from the vessel A through the siphon.

In passing from A to C the liquid loses potential energy, being carried from a higher to a lower level, and hence this motion takes place.

It is clear from the proof that, if C be above D, the level of the water in the vessel, the siphon will not work; on opening the end C the water in the tube would flow back into the vessel.

Again, it is also clear that the height BD must not be greater than the barometric height of the liquid to be emptied. For the column BD is supported in the tube by the atmospheric pressure on its base; if then BD be greater than the barometric height, some of the liquid will run back into the vessel A until the height of the column of liquid in BD is just equal to the barometric height. In this case, when C is opened, the liquid in BC will run out until the upper free surface of the column in BC is at the barometric height above C.

The final condition will depend on various circumstances; it is quite possible that the momentum acquired by the column in BC may be sufficient to carry the whole of the column out of the tube, or again some portion of the column may run out and then the atmospheric pressure acting on the lower end of the column may force it back up the tube until it joins the column in the other limb and then the whole of the liquid in the tube will flow back into the vessel A.

For convenience in filling, the siphon is often made in the form shewn in Fig. 80, the lower end is closed and the liquid is drawn from the vessel by suction at the side tube.

87. Experiments with the Siphon.

An experiment due to Pascal illustrates the action of the siphon. A three-armed glass tube of the form shewn, ABC,

Fig. 81, is employed. The two lower arms dip into two vessels of mercury, the arm C being longer than the arm A.

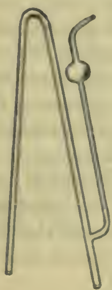


Fig. 80.

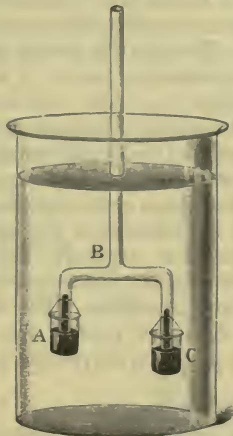


Fig. 81.

The whole apparatus is immersed in a deep vessel of water, the arm B being kept open to the atmosphere. As the tubes are lowered, the water pressure forces the mercury up the tubes A and C until the column in the shorter tube rises to its top, runs along the horizontal tube and joins the column in the tube A; thus a continuous column of mercury is formed from A to C. When this has taken place the tube A to C forms a siphon and the mercury flows from A to C.

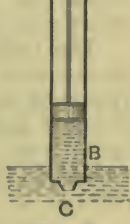


Fig. 82.

88. The Syringe.

This instrument is the simplest form of pump for raising water. It consists of a hollow cylinder AB, Fig. 82, in which a solid air-tight piston works; the lower end of the cylinder terminates in a nozzle C, which is placed under the liquid it is desired to raise.

Let the syringe, with the piston at the bottom

of the tube, be placed in the liquid; if the piston be now raised, the atmospheric pressure, acting on the upper surface of the liquid, forces it through the nozzle into the cylinder to fill the vacuous space which would otherwise be formed under the piston. Thus as the piston is raised the liquid flows into the cylinder after it.

The syringe and the various forms of pump act by the principle of suction; this consists in enlarging the volume of a space to which the liquid has access; the pressure within the space is thus reduced and the atmospheric pressure forces the liquid into the space until equilibrium is again established. In this way air is sucked into the lungs; the muscles of the chest cause the lungs to expand; the internal pressure is thereby reduced and the air passes in. The act of drinking water through a tube is similar. The drinker causes the air in his mouth and the upper part of the tube to expand and then the atmospheric pressure drives the water up.

89. Valves.

In most hydrostatic machines valves are employed. A valve may be described as a trap-door which will open in one direction only; it will thus yield to an excess of pressure on one side and, opening, will allow the passage of fluid; an excess of pressure in the other direction will close the valve and stop the flow of the fluid.

A simple form of valve is the hanging flap valve; it is a flat disc which turns about a hinge in its upper edge and thus opens or closes a passage. In the ordinary bellows the flap is a piece of leather; this is raised when the bellows are expanded and allows the air to enter. On compressing the bellows the leather disc is driven against the opening it covers and the air is forced through the nozzle.

The ball-valve is another form in use. A metal sphere fits accurately over the opening of a pipe through which the fluid is to pass, when fluid is forced along the pipe the sphere is raised and it passes out; pressure in the other direction only drives the sphere more closely against its seat. The sphere is constrained by suitable guides, so that it can only rise and fall and not move far from the orifice it is to close.

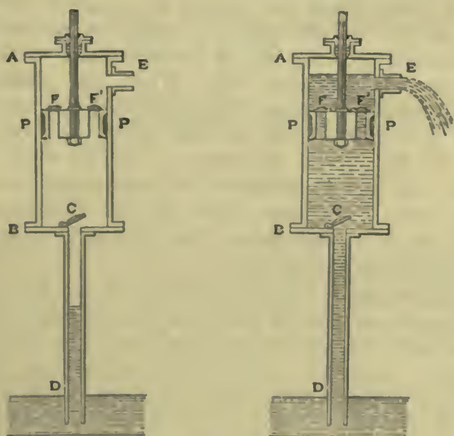
A form of valve used in many air-pumps is shewn in

Fig. 91 (a). It consists of a strip of oiled silk covering tightly a small orifice; two opposite sides of the strip are secured firmly, the other two are free; when air is forced against the valve through the orifice the silk stretches slightly and the air escapes under the free edges of the silk; if air is forced in the other direction the silk is drawn against the orifice and closes it securely.

A perfect valve would work with the very slightest difference of pressure; in reality no valve satisfies this condition; a definite excess of pressure is required to work it and there is always some leakage.

90. The common Pump.

This consists of a cylinder AB, Fig. 83, in which a piston P works. The piston is fitted with a valve F opening



(i) Fig. 83. (ii)

upwards. (In the figure there are two such valves.) Above the piston is the spout E from which the water flows; the bottom of the piston is connected to a tube CD which communicates with the water to be raised. This tube is closed by a valve C opening upwards.

Let the piston be at the bottom of the cylinder and suppose

that the tube CD and the cylinder contain no water but are filled with air. Raise the piston; this increases the space below the piston and reduces the pressure; the atmospheric pressure acting on the valves F, F' closes them; the pressure of the air in DC becomes greater than the pressure above the valve C. Thus this valve is raised and the air in DC expands into the part of the cylinder below the piston. In consequence the pressure on the surface of the water in the tube at D is reduced and the atmospheric pressure forces the water up the pipe DC.

Now depress the piston, the pressure in the cylinder is increased and the valve C is in consequence closed. The valves F, F' open when the pressure below the piston becomes greater than that above and the air escapes; this continues until the piston reaches the bottom of the cylinder. When it is again raised the process is repeated and the water is drawn further up the pipe DC, until at last after several strokes, depending on the size of the pump and of the pipe, the water enters the cylinder AB. When the piston is again lowered the water is forced through the piston valves and is raised at the next stroke to the spout E, from which it escapes. In Fig. 83 (i) the pump is shewn with the piston rising after one or two strokes; the water is still in the pipe only, the valve C is open while F and F' are closed; in Fig. 83 (ii) it is in full action; the piston is rising and the water issuing from the spout.

Since the water is raised in DC solely by the atmospheric pressure it is clear that, if the pump is to work, DC must not exceed the height of the water-barometer, otherwise the water would never reach the valve C and enter the cylinder; the sole result of the pumping would be to maintain a column in the pipe DC equal in height to that of the water-barometer.

It was the failure of certain pumps belonging to the Grand Duke of Tuscany to act which led Galileo in about 1640 to investigate the pressure of the atmosphere.

The following Examples will illustrate the action of a pump.

Examples. **(1)** *To find the force on the piston rod required to raise the water in a common pump*[1].

[1] The force actually exerted will be very considerably greater than that calculated in this example, because of the friction.

(i) *Before the water has risen to the spout.*

Let the height of the water-barometer be H cm., the height of the column in the pipe DC h cm., the pressure of the air in the cylinder, measured as a head of water, p, and the area of the piston A sq. cm.

The force on the upper side of the piston is the weight of $A \cdot H$ c.cm. of water, that on the lower face is the weight of $A \cdot p$ c.cm. of water.

Hence the force required is $A(H-p)$[1] grammes weight.

But the pressure on the top of the column in the pipe is p, that on the bottom is H.

Hence $$p+h=H.$$

Therefore $$H-p=h.$$

Thus the force on piston rod $=A \cdot h$ grammes weight $=$ weight of a column of water equal in cross section to the area of the piston and in height to that of the column in the pipe.

(ii) *When the pump is in full work.*

Let the piston be at a depth k below the spout, the other symbols being as before.

The downward force is $A(H+k)$, the upward force is $A \cdot p$.

Hence the force required is $A(H+k-p)$.

But as before $$p=H-h.$$

Thus the force required $=A(h+k)=$ weight of a column of water equal in cross section to the area of the piston and in height to that of the spout above the water in the well.

(2) *Find the height the water rises in one stroke.*

(i) *When the water is below the lower valve C.*

Let a be the length of the stroke, C the area of the pipe and c the height of the bottom of the cylinder above the water in the well; let h be the height of the water in the pipe above the well at the beginning of the stroke, z the distance it rises during the stroke. Then at the beginning of the stroke the air above the water occupies a volume equal to $C(c-h)$ and its pressure is $H-h$. At the end of the stroke the volume is $C(c-h-z)+A \cdot a$ and the pressure is $H-h-z$.

Hence, since the product of the volume and pressure is equal,

$$(H-h)\{C(c-h)\}=(H-h-z)\{C(c-h-z)+A \cdot a\},$$

and from this equation z can be found, and this is the height the water rises.

(ii) *When during the stroke the water rises into the cylinder.*

Let y be the depth of water in the cylinder at the end of the stroke. At the beginning the water was at a depth $c-h$ below the valve. The

[1] If we are not working in centimetres and grammes but in some other units, the force will be $\omega A(H-p)$, where ω is the weight of a unit of volume of water.

volume of the air before the stroke was $C(c-h)$ and its pressure $H-h$. After the stroke the volume of the air is $A(a-y)$ and the pressure is $H-(c+y)$.

Hence, since the product of the volume and pressure is constant,

$$(H-h)\,C(c-h)=\{H-(c+y)\}\,A(a-y).$$

From this equation y can be found and then the rise is $y+c-h$.

91. The Lift-pump.

This is shewn in Fig. 84, and consists of a cylinder AB with a valve C at the bottom opening upwards. From C a pipe CD leads to the well. A piston P works in the cylinder, and in the piston there is a valve G, also opening upwards. A tube EF communicates with the upper part of the cylinder and is closed by a valve at E opening outwards from the cylinder.

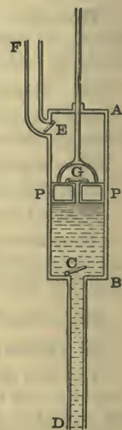


Fig. 84.

Let the piston be at the bottom of the cylinder and suppose the pipe CD and the cylinder are filled with air. Raise the piston. The volume of the space below the piston is thus increased and the pressure in it is diminished; the atmospheric pressure acting on the valve G closes it; the pressure of the air in DC becomes greater than the pressure above the valve C. Thus the valve is raised and the air in DC expands into the part of the cylinder below the piston. In consequence the pressure of the water in the pipe at D is reduced and the atmospheric pressure forces the water up the pipe DC. At the same time the air in the cylinder above the piston is compressed and in consequence the valve E is opened; the air then escapes up the tube EF.

Now depress the piston; the pressure in the cylinder is increased and the valve C is, in consequence, closed. The valve G opens and the air from below passes into the cylinder above the piston. This continues until the piston reaches the bottom of the cylinder. When it is again raised the process is repeated and the water forced further up the pipe DC until at last the water enters the cylinder AB. When the piston

is again lowered the water passes through the piston-valve and is raised at the next stroke to the valve E. The pressure opens this valve and the water passes on up the tube EF. When the piston is again depressed the valve E is closed by the water-pressure above and remains closed until the next upstroke, when it is again opened and more water is lifted up the tube EF. The process thus continues; the height to which the water can be lifted depends only on the force applied to the pump-handle and the strength of the pump.

The force on the piston until the water has reached the tube EF is given by the same expression as that found for the common pump. When the water has been raised into the tube EF, let h' be the head of water above the piston. Then the force on the upper side of the piston is $(H+h')A$ grammes weight, that on the lower side acting upwards is $(H-h)A$ grammes weight, h being the height of the piston above the well.

Thus the resultant downward force is

$$(H+h')A-(H-h)A \text{ or } (h'+h)A \text{ grammes weight.}$$

It is thus the weight of a column of water, having the area of the piston for its base and the total height to which the water has been raised for its height.

The rise of the water for a single stroke can also be found.

92. Force-pump.

This consists of a cylinder AB with a solid piston or a plunger H, working in an air-tight collar, as in Fig. 85.

At the bottom of the cylinder there is a valve F opening upwards. A tube CE rises from close to the bottom of the cylinder and is closed with a valve C opening outwards from the cylinder.

Suppose that the piston is initially at the bottom, let it be raised; the pressure in the cylinder is diminished, the atmospheric pressure closes the valve C, the valve F is opened and air passes in from the pipe BD, thus reducing the pressure in the pipe at D. The atmospheric pressure forces the water up the pipe DB. As the piston descends the valve F is closed while C is opened and the air is forced from the cylinder up the tube CE.

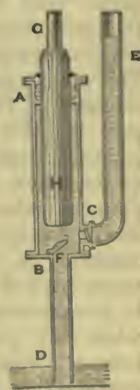


Fig. 85.

This process is repeated until the water rises into the cylinder above the valve F; at the next downstroke the water is forced through the valve C and up the tube CE. When the piston again rises the water-pressure in CE closes the valve C. More water is forced up the pipe through the valve F to be forced up the tube on the next downstroke. Thus the height to which the water can be forced depends on the force applied at the handle and the strength of the pump.

The force on the piston rod during the upstroke is found in the same way as for the common pump, Section 90.

During the downstroke the force will depend on the height to which the water has been raised in the tube CE. This force must clearly be applied downwards, and if A be the area of the piston, h' the height of the water in CE above the bottom of the piston, the force is Ah' grammes weight.

In all three pumps it is necessary that the height of the valve at the bottom of the cylinder above the water in the well should be less than the height of the water-barometer; in practice it is found that this height must be considerably less because of the imperfection of the valves.

93. Continuous action pumps. The fire-engine.

In the pumps just described the action is discontinuous; in the first two, the water only flows from the spout during the upstrokes; in the last the flow occurs only during the downstroke. In some pumps this is remedied by the introduction of an air chamber. The water is forced into a closed chamber A, Fig. 86. From the lower part of this chamber a pipe BD passes through the top of the chamber to the spout. When the water is first pumped into the chamber it rises above the open end of the pipe BD, enclosing a quantity of air in the upper part of the chamber. This air is compressed by the water which, when the piston is being rapidly lowered, enters the chamber with considerable velocity. As the piston is raised, and the valve C is closed,

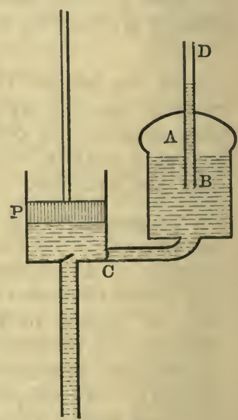


Fig. 86.

the air in *A* expands, thus driving the water, which otherwise would cease to flow, up the tube *BD*. Energy is stored up in the compressed air during the downward stroke, this is used to raise the water during the upward stroke.

The hand fire-engine consists of two force-pumps connected to a common air chamber. The handles of the pumps are so arranged that while one descends the other rises.

94. Bramah's Press.

The principle of this apparatus has already been described as an illustration of the transmissibility of pressure and was known to Pascal. Two cylinders filled with fluid and fitted with pistons of different areas are connected together, thus a small force applied to the small piston enables the large piston to exert a much greater force.

The invention was however useless for very many years because of the difficulty of rendering watertight the apertures through which the piston rods work. Bramah overcame the difficulty by his invention of the cupped leather collar. A leather ring, semicircular in section, fits round the pistons in a groove in the sides of the cylinder. The concavity of the ring is turned downwards and water passing between the

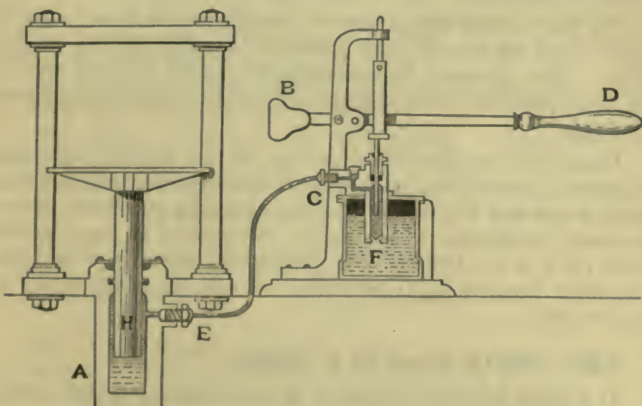


Fig. 87.

piston and the sides of the cylinder fills the hollow under the ring, and by its pressure forces the ring against the sides of the piston, thus forming a packing which becomes more tight as the pressure increases.

The press is shewn in Fig. 87 and the collar in Fig. 87 *a* A small solid plunger is worked by the handle D; this as it rises draws liquid through a valve shewn at F from a reservoir. When the plunger descends the liquid is forced into the large cylinder A through a valve opening into the tube CE. The pressure at each point of the liquid in A is then the same as in the small cylinder, and since the area of the plunger H is much greater than that of the other plunger, the upward thrust exerted on H is greater than the downward thrust exerted by the small plunger. The force is applied to the small plunger by the lever BD. By shifting a pin on which this lever works, its shorter arm can be considerably reduced and greater mechanical advantage obtained

Fig. 87 *a*.

Let P be the force applied to the lever, and m its mechanical advantage. Let a be the area of the small plunger, A that of the large.

The downward thrust on the small plunger is mP, this is applied over an area a; the pressure therefore in the fluid is mP/a. This pressure is transmitted to each unit of area of the large plunger; thus the upward thrust on the large piston is mPA/a.

Hence the mechanical advantage of the whole instrument is mA/a. Thus if the long arm of the lever be 10 times that of the short, and the area of the large plunger 100 times that of the small, conditions which might easily occur in practice, the value of A/a is 100 and the mechanical advantage is 1000. Any force applied at D is multiplied one thousand times by the machine.

*95. Work done in a Press.

It follows from the principle of work that, in order to raise a weight through a given distance by such a press, the small

plunger will have to move over 100 times that distance and the handle of the lever over 1000 times the distance.

We can easily shew that this is true thus. Let the end of the small plunger descend a distance x while the large plunger rises a distance X.

Then the volume of the water in the small cylinder is decreased by $a \,.\, x$, that in the large cylinder is increased by $A \,.\, X$. But these expressions must be equal since each is the volume of water forced from one cylinder to the other.

Therefore $$AX = ax.$$

Thus $$X = x\frac{a}{A} = \frac{x}{100}$$

if the ratio of the areas of the plungers be 100 to 1.

Again, if y represent the distance traversed by the handle when the plunger descends a distance x, then $y = mx$.

Hence $$X = x\frac{a}{A} = y\frac{a}{mA} = \frac{y}{1000}$$

if $A/a = 100$ and $m = 10$.

Examples. (1) *The lower valve of a common pump is* 10 *feet above the water and the area of the cylinder is four times that of the pipe leading to the well. Assuming the height of the water-barometer to be* 33 *feet, find the length of stroke if the water just rises to the valve at the end of the first upstroke.*

Let the stroke be x feet and let w be the weight of a unit of volume of water and a the area of the lower pipe.

The volume of the air in the pipe originally is $10a$ c. feet, and its pressure is $33w$.

After one stroke its volume is $x \times 4a$ c. feet, and its pressure $(33 - 10)w$.

Now the product of the volume and pressure is constant by Boyle's Law.

Hence $$10a \times 33w = 4xa \times 23w.$$

Therefore $$x = \frac{10 \times 33}{4 \times 23} = 3{\cdot}59 \text{ feet.}$$

(2) *A force-pump is used to draw water from a depth of* 5 *metres and drive it to a height of* 20 *metres, the diameter of the plunger is* 25 *cm.; find the force on the piston rod in the back and forward stroke.*

The area of the plunger is $\frac{1}{4}\pi \times 625$ sq. cm.

During the back stroke the pressure on the piston is less than that outside by that due to 5 metres of water. The force on the piston therefore is $500 \times \frac{1}{4}\pi \times 625$ grammes weight[1] or $2{\cdot}454 \times 10^6$ grammes weight.

In the forward stroke the pressure in the cylinder exceeds that outside by that due to 20 metres of water. The force therefore is four times as great as previously and acts in the other direction.

It is therefore $9{\cdot}816 \times 10^6$ grammes weight.

(3) *The stroke of a common pump is* 8 *inches, the diameter of the barrel* 4 *inches, that of the pipe* 1 *inch. The lower valve is* 15 *feet above the reservoir. How high does the water rise on the first stroke, assuming the height of the water-barometer to be* 30 *feet?*

Let the water rise x feet.

Initially the volume of air in the pipe is $\frac{1}{4}\pi \times 1^2 \times 15 \times 12$ c. inches, and its pressure is that due to 30 feet of water. At the end of one stroke the volume is $\frac{1}{4}\pi\{1^2 \times (15-x)12 + 16 \times 8\}$ c. inches, and the pressure is that due to $30-x$ feet of water. Therefore by Boyle's Law,

$$\tfrac{1}{4}\pi \times 15 \times 12 \times 30 = \tfrac{1}{4}\pi\{(15-x)\,12 + 16\,.\,8\}\,(30-x).$$

Therefore $$15\,.\,3\,.\,30 = \{(15-x)\,3 + 4\,.\,8\}\,(30-x).$$

Hence $$3x^2 - 167x + 960 = 0$$

and $x = 6{\cdot}5$ approximately.

Thus the water rises about 6·5 feet.

(4) *In a Bramah press the diameter of the ram is* 20 *inches, that of the piston* 2·5 *inches. What force must be applied to the piston to raise* 1000 *tons weight?*

The areas of the ram and the piston are $\frac{1}{4}\pi \times 400$ and $\frac{1}{4}\pi \times 6{\cdot}25$ square inches respectively.

Thus the force required is $1000 \times 6{\cdot}25/400$ or 15·625 tons weight.

(5) *If* A *be the area of the cross section of the piston of a force-pump,* l *the length of the stroke,* n *the number of strokes per minute and* B *the area of the pipe from the pump. Find the mean velocity with which the water runs out.*

At each stroke a volume $A\,.\,l$ of water is transferred to the pipe, the length of this when in the pipe is Al/B. Hence the length of the column transferred to the pipe per minute is nAl/B. Hence a particle which was at the valve when a stroke began has at the end of 1 minute moved a distance nAl/B. This length then measures its velocity per minute.

[1] Strictly the force on the piston will alter slightly with its position in the cylinder; it is assumed that the length of stroke may be neglected compared with the heights the water is raised.

(6) *Determine the work done in each stroke of a common pump after the water has risen to the spout.*

If h is the height of the spout above the well, A the area of the piston and l the length of the stroke, the force on the piston is the weight of a volume Ah of water; hence if ω be the weight of a unit volume of water the force on the piston rod is ωAh, but the piston rod moves a distance l, hence the work done is ωAhl.

If we write this $\omega Al \times h$ we see that it is the work done in raising a volume of water which would fill the cylinder to the height of the spout.

This result is obvious.

(7) *Find the work done in a complete stroke back and forwards of a force-pump.*

(i) When the stroke has been completed the state of the piston and of the water in the cylinder is exactly as before, but a volume of water Al, where A is the area of the piston and l the length of the stroke, has been raised a height $h+h'$, where h' is the height of the spout above the lower valve and h that of the lower valve above the well; thus the work done is $\omega Al(h+h')$.

Otherwise thus:

(ii) Let h be the height of the lower valve from the well, h' the head of water in the delivery tube, and let the piston be at a height x from the bottom of the cylinder. When it is descending the force on it is $\omega A(h'-x)$ upwards, hence in descending a small distance δ through this position the work done is $\omega A(h'-x)\delta$.

When the piston is in the same position but ascending the force is $\omega A(h+x)$ and the work is $\omega A(h+x)\delta$.

Hence the total work done in traversing the distance δ down and up is $\omega A(h'-x)\delta+\omega A(h+x)\delta$ or $\omega A(h+h')\delta$.

Thus the work done in traversing the whole stroke l down and up is $\omega A(h+h')l$ as before.

It follows also from this that the work done in descending is $\omega A(h'-\frac{1}{2}l)l$, that in ascending is $\omega A(h+\frac{1}{2}l)l$.

We may obtain these results otherwise as follows:

(iii) In raising the piston a volume of water equal to the volume of the cylinder is lifted and the centre of gravity of this volume is raised to a height $(h+\frac{1}{2}l)$. Hence the work done is $\omega Al(h+\frac{1}{2}l)$. In depressing the piston this same volume is raised to a height h' above the bottom of the cylinder, that is to a height $h'-\frac{1}{2}l$ above the original position of its centre of gravity.

Thus the work done is $\omega Al(h'-\frac{1}{2}l)$.

96. Hawksbee's Air-pump.

This consists of a cylinder called the barrel AB, Fig. 88,

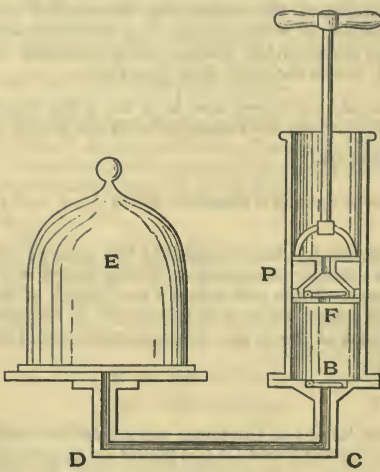


Fig. 88.

in which a piston P works. This piston has a valve F opening upwards; at the bottom of the cylinder is another valve B also opening upwards and closing a pipe CD which leads from E to the receiver or vessel to be exhausted.

On raising the piston the pressure in the lower part of the barrel is reduced; the air in the receiver expands, opening the valve B, and passes into the barrel. Thus the pressure in the receiver is reduced. When the piston is depressed the valve B is closed by the increasing pressure in the barrel, the piston valve F is opened and the air which during the upstroke was withdrawn from the receiver passes through it and escapes. Thus at each stroke the air which fills the barrel is withdrawn. The exhaustion however is never complete, for there is always some space, known as the "clearance," left at the bottom of the barrel even when the piston is pushed quite home. This space is then filled with air at atmospheric pressure; when the piston is at the top of its stroke therefore there will be air at a small

pressure inside the barrel, and since the air in the receiver escapes into the barrel by raising the valve B its pressure can never be less than this limiting pressure of the air in the barrel.

When the piston is being raised the downward thrust which is being overcome is that due to the atmospheric pressure, the upward thrust is that due to the pressure of the air in the receiver; when the exhaustion is considerable the difference between these two will be great and towards the end of the exhaustion the full atmospheric pressure of about 1 kilogramme weight per square centimetre of the piston has to be overcome; when the piston is descending this same force presses it down. Thus a considerable amount of energy is spent uselessly in securing the exhaustion.

97. Smeaton's Air-pump.

This consists of a cylinder or barrel in which a piston with a valve opening upwards works; the cylinder is closed at both ends. One end communicates with the receiver through a tube, and the tube is closed by a valve opening into the cylinder, the other communicates with the atmosphere through a valve opening outwards[1].

On raising the piston, the piston valve is closed and the air above forced out through the upper valve. The pressure below the piston is diminished and the air from the receiver opens the lower valve and expands into the barrel. On lowering the piston both the cylinder valves are closed, the piston valve is opened and the air below the piston passes above. At the next stroke this air is expelled through the upper valve, and more air is drawn from the receiver to fill the barrel; the process is then repeated.

This pump has two advantages over Hawksbee's. In the first place until the upper valve is opened the pressure on the upper side of the piston during the upstroke is less than the atmospheric pressure; thus less force is required to raise the piston; while during the downstroke the difference of pressure between the two sides of the piston is very small.

Secondly, when the piston is at the bottom of the barrel after a few strokes the pressure of the air above the piston is very small instead of being as in Hawksbee's pump the atmospheric pressure; hence the pressure of any air which is left in the clearance space below the piston is small and the exhaustion can be made more complete.

[1] The pump is thus the same as Hawksbee's except that the communication with the atmosphere is through the last valve.

98. Double-barrelled Pump.

This is shewn in Fig. 89. In the ordinary pump no air is exhausted during the downstroke; in the double-barrelled

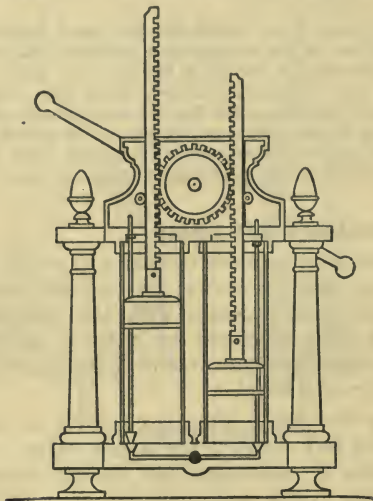


Fig. 89.

pump there are two cylinders and pistons. These are connected together by racks and a pinion, so that while one piston is rising the other is falling; thus air is exhausted during each stroke. Moreover since the atmospheric pressure tends to depress each piston equally, its effect on the one piston which is rising is just balanced by that on the other, which is descending. Less force therefore is necessary to raise the piston than in the ordinary pump; this compensation, it should be observed, does not exist throughout the stroke, for the pressure under the descending piston is continually increasing and when the piston-valve is open the atmospheric pressure on the other piston is entirely unbalanced. As the exhaustion proceeds the compensation at all parts of the stroke becomes more complete, and the pump is easier to work.

99. Tate's Air-pump.

This, which is a very usual form, is shewn in Fig. 90, and in section in Fig. 91. A double piston, P, P', works in the

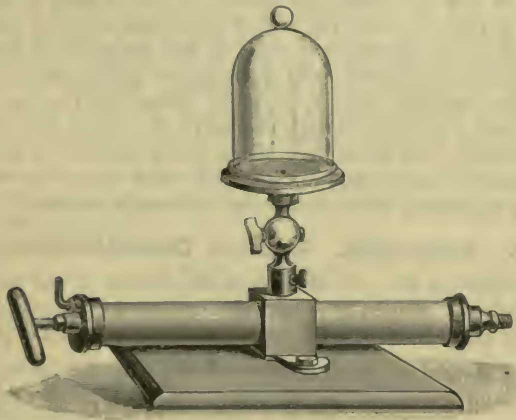


Fig. 90.

barrel AB[1]. At A and B are valves opening outwards. The construction of these is shewn in Fig. 91 (a). The pipe

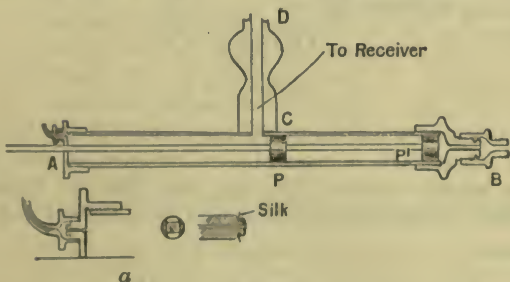


Fig. 91.

CD leads to the receiver. In the figure the portion AP of the barrel is in communication with the receiver.

[1] In some cases the piston is a solid plunger.

When the piston is drawn back the air in AP is compressed, the valve at A is opened by the pressure and the air is expelled. The distance between the pistons is such that, when P is brought home, the piston P' just comes to the left of CD, so that the receiver is now in communication with the other end of the barrel, and some of the air which it contains expands into the barrel. On pushing the piston back the air which has entered BP' is shut off from the receiver, compressed and expelled through the valve B. Thus the receiver is exhausted. The advantage of the pump lies in the fact that the valves in the piston and in the tube from the receiver are both dispensed with, hence the leakage is reduced.

100. Air-pumps. General Considerations.

PROPOSITION 28. *To determine the density of the air, after any number of strokes of a single barrel air-pump, in terms of its initial value and of the volumes of the barrel and the receiver.*

Let V be the volume of the receiver, v of the barrel, ρ the original density, $\rho_1, \rho_2 \ldots \rho_n$ the density after 1, 2...n upstrokes.

After one upstroke the air in the receiver occupies the receiver and the barrel; its volume therefore changes from V to $V+v$ and its density from ρ to ρ_1. The mass of air however is unchanged and the mass is equal to the product of the volume and the density.

Thus $$(V+v)\rho_1 = V\rho$$

or $$\rho_1 = \frac{V}{V+v}\rho.$$

On the next downstroke the air is divided into two parts, a mass $v\rho_1$ escapes, a mass $V\rho_1$ is retained in the receiver. After a second upstroke this last mass fills the receiver and the barrel; its density is then ρ_2 and its volume $V+v$.

Hence $$(V+v)\rho_2 = V\rho_1.$$

Therefore $$\rho_2 = \frac{V}{V+v}\rho_1 = \left(\frac{V}{V+v}\right)^2\rho.$$

Proceeding thus we see that the density after any stroke is found by multiplying the density before the stroke by the proper fraction $V/(V+v)$.

Thus, after n strokes, we have

$$\rho_n = \frac{V}{V+v} = \rho_{n-1} = \ldots\ldots = \left(\frac{V}{V+v}\right)^n \rho.$$

Hence, when n is large, the density is very considerably diminished and, if the law were to hold continuously, could be made as small as we please by sufficiently increasing the number of strokes. As we have seen however in practice, this condition cannot be realized because of the necessity of leaving clearance spaces and of the imperfection of the valves.

Since the pressure of air at a constant temperature is always proportional to its density we may write in place of the above equation

$$p_n = \left(\frac{V}{V+v}\right)^n p,$$

where p_n is the pressure after n strokes and p the original pressure.

101. Measurement of the Pressure of the Air in a Receiver.

The pressure of the air in a receiver is measured experimentally by the use of one or other of the gauges described in Sections 36–39. For fairly high exhaustions a vacuum siphon gauge, Fig. 31, is usually employed. If it be desired to watch the effect of each stroke on the pressure, the vertical tube shewn in Fig. 32 is convenient. The top of the tube is connected with the receiver and the height of the column is measured after each stroke. The differences between the height of the barometer and these heights give the pressures.

Examples. (1) *The capacity of the barrel of a Smeaton's air-pump is $\frac{1}{9}$th of that of the receiver; determine the pressure after five strokes.*

Let the volume of the barrel be v, then that of the receiver is $9v$. Thus a mass of air which occupies $9v$ c.cm. before any stroke occupies $10v$ c.cm. at the end of the stroke. At each stroke therefore the pressure is reduced in the ratio 9 to 10. Thus after five strokes it becomes $(\frac{9}{10})^5$ of its initial value.

Now $(\cdot 9)^5$ is approximately $\cdot 59$. Hence the pressure is reduced to about $\cdot 59$ of its original value.

(2) *How many strokes must be made with the same pump to reduce the pressure to ·1 of its original value?*

Let the number of strokes be n, then after n strokes the initial pressure p is reduced to $(\cdot 9)^n p$.

Hence $$\cdot 1 = (\cdot 9)^n.$$

We can find a value for n by trial.

Thus for $n=5$, $(\cdot 9)^n = \cdot 59$.

Hence $n=10$, $(\cdot 9)^n = (\cdot 59)^2 = \cdot 35$ approximately,

$n=20$, $(\cdot 9)^n = (\cdot 35)^2 = \cdot 12$ approximately.

Thus for $n=21$, $(\cdot 9)^n = \cdot 12 \times \cdot 9 = \cdot 108$,

$n=22$, $(\cdot 9)^n = \cdot 108 \times \cdot 9 = \cdot 0972$.

Hence during the twenty-second stroke the pressure will reach the required value.

But we can find the result more readily by the use of logarithms, thus

$$\cdot 1 = (\cdot 9)^n,$$

$$\log (\cdot 1) = n \log (\cdot 9),$$

$$n = \frac{\log \cdot 1}{\log \cdot 9} = \frac{-1}{-\cdot 046} = \frac{1000}{46} = 21\cdot 16,$$

giving the same result as above, viz. that during the twenty-second stroke the required value is attained.

(3) *The capacity of the receiver of a Smeaton's pump is nine times that of the barrel. At what point in the sixth upward stroke will the upper valve open?*

When the piston is at the bottom, after five upward strokes, the pressure of the air in the barrel is ·59 of the atmospheric pressure (see Example 1).

As the piston rises during the sixth stroke the air is compressed, and it must be compressed to ·59 times its original volume to increase the pressure up to that of one atmosphere. Thus the piston must complete $1 - \cdot 59$ or ·61 of its stroke.

(4) *In one air-pump the barrel has $\frac{1}{10}$th of the volume of the receiver, in another it has $\frac{1}{5}$th. How many strokes of the latter are needed to produce the same exhaustion as that due to four of the former?*

In the first pump the density is reduced by each stroke in the ratio 10 to 11. Thus after four strokes the density becomes $(10/11)^4$ or ·682 of its former value.

In the second pump the reduction for each stroke is 5/6 or ·833. Thus after two strokes the density becomes $(\cdot 833)^2$ or ·694 of its former value and hence two strokes of the second are rather less effective than four of the first.

102. Mercury Air-pumps.

When working with the air-pumps which have been described it is impossible, because of leakage and the necessity for clearance space, to secure a very high vacuum. Various forms of mercury-pump have however been invented, in which the exhaustion can be carried to a much higher degree of exhaustion. Sprengel's pump and Geissler's pump well illustrate two types of mercury-pump.

*103. Sprengel's Air-pump.

This consists of a vertical glass tube BC, Fig. 92, the lower end of which dips under mercury in a vessel G. The upper end is connected with a reservoir E which can be filled with mercury. At B a branch tube, BD, is inserted; this communicates with the vessel to be exhausted. The height, BG, is greater than that of the mercury-barometer. The reservoir E is usually connected with the vertical tube by a short piece of flexible tubing which can be closed by a clamp F. Let us suppose also that it is possible to close the tube to the receiver, by a tap or second clamp D, and let this tap be closed. Release the clamp F; mercury will run down the tube BC carrying the air before it and will completely fill the tube. Close the clamp F. The atmospheric pressure will sustain a column of mercury equal in height to that of the barometer in the tube BC. Above this column there will be a vacuum. Now open D. Air from the receiver expands into this vacuum, the mercury column falls somewhat and the pressure in the receiver is reduced. Close D and open F; the stream of mercury down BC carries the air again out of that tube and, on closing F, the mercury stands at the barometric height. Again open D; more air enters BC from the receiver and this, on closing D and opening F, can again be removed. This description may serve to explain the action of the pump, but, in practice, the tap D is unnecessary and

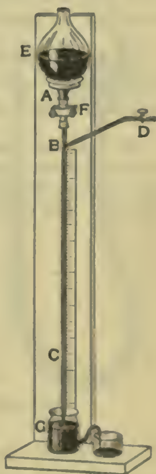


Fig. 92.

the clamp F is only used to stop the flow when the receiver is sufficiently exhausted. It is found that if the receiver is permanently connected to the vertical tube BC and the clamp F is opened, the process described goes on in a partial manner continuously. The mercury column descending from E breaks into drops at B; as the pressure in BC is reduced, through the air being carried down by the mercury, the air from the receiver expands into the tube and is carried down between the drops. There is no need therefore alternately to open and close the tap D; hence it may be removed and the risk of leak which it gives rise to may thus be avoided. The process continues until the degree of exhaustion in the receiver is comparable with that of a Torricellian vacuum.

As the mercury from the reservoir E flows away, it is replaced by mercury overflowing from G, which is caught in a suitable vessel at one side.

In the more modern forms of Sprengel's pump, which are used for exhausting the bulbs of incandescent lamps and other work needing a very high exhaustion, the tube ABC is bent in the manner shewn in Fig. 93. The efficiency of the pump is thereby increased.

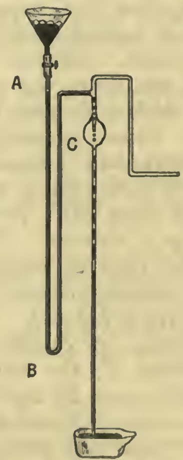


Fig. 93.

*104. Geissler's Air-pump.

This pump in its simplest form consists of two glass reservoirs of considerable capacity. One of these *A*, Fig. 94, is fixed, the other, *B*, can be raised or lowered at will; the two are connected by a piece of stout india-rubber tubing.

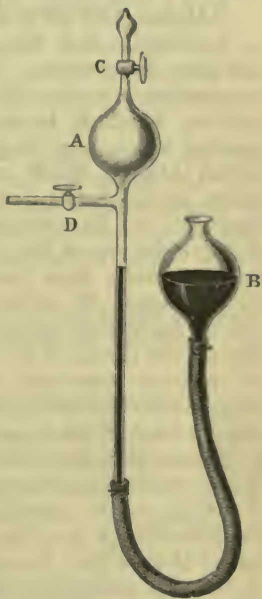


Fig. 94.

The moveable reservoir *B* is open to the atmosphere, while *A* can be put into communication with the atmosphere through a tube which enters at its top; the tube is closed by a tap *C*. A second tube, connected to *A*, communicates with the receiver to be exhausted through a pipe, which can be closed by a second tap *D*. The reservoir *B* when in its lowest position is filled with mercury. Close the tap *D* and open *C*. Raise *B* slowly until it is at a slightly higher level than *A*. The mercury passes from *B* into *A*, driving the air out until *A* is filled with mercury up to the level of the tap *C*, all the air being expelled. Close the tap *C* and lower *B*; the mercury passes back into *B* and stands at the barometric height in the tube between the two reservoirs. Now open the tap *D*. The air from the receiver expands into *A* and the pressure in the receiver falls. Close *D* and raise the reservoir *B*, the air which now fills *A* is compressed by the mercury as it rises into *A*; when the mercury is nearly at the same level in the two, indicating that the air above the mercury in *A* is at atmospheric pressure, open the tap *C* and continue to raise *B* until the mercury in *A* rises again to *C* Then close *C* and repeat the process. In this way the air is drawn from the receiver and a high degree of exhaustion is attained.

In the form of pump just described the tap D is a source of difficulty. It is nearly impossible to make the fit so good that there shall be no leak when the tap is turned, and thus the vacuum which can be obtained is impaired.

*105. Topler's Air-pump.

In Topler and Hagen's modification, Fig. 95, the taps are done away with. Communication is made with the receiver which is to be exhausted through a long inverted U-tube HDE which enters at the bottom of the reservoir A. The height of D above the top of the reservoir is greater than that of the barometer. To reduce the risk of fracturing the pump, by a sudden inrush of air from the receiver when the mercury in the reservoir falls below the level of the tube DH, a side tube connects C the top of the reservoir to the point H just above the junction of the reservoir A and the inverted U-tube DH. From the top of the reservoir a tube CFG bent twice at right angles runs downwards and ends, either under mercury in a small vessel, or in a siphon bend as shewn in the figure. The distance of the bottom of this bend below C is greater than the barometric height.

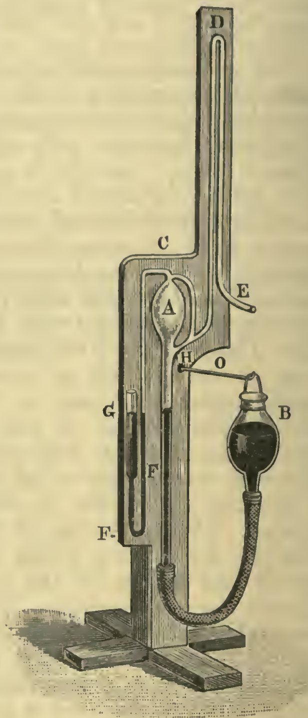


Fig. 95.

As the vessel B is raised the mercury in its ascent closes the mouth of the tube HDE and thus shuts off the air in the receiver from that in the reservoir A. The air in A is then compressed, and finally by raising B until the mercury begins

to flow over into CFG it is expelled through this tube. The vessel B is then lowered. The atmospheric pressure forces mercury from the siphon bend up FC, but since CF is greater than the barometric height the mercury does not reach C and the reservoir A is completely shut off from the atmosphere. A vacuum is thus formed in the reservoir A until the level of the mercury it contains falls below the point H where the tube from the receiver enters, when this is the case, air enters from the receiver. The process is then repeated.

It is necessary that DH should be greater than the barometric height for, as the vessel B is raised, the air-pressure on the surface of the mercury in B forces mercury up the tube DA and, when the exhaustion is considerable, this mercury will rise to very nearly the barometric height above the level of C.

In using the pump the vessel B can be raised and lowered by hand, in general however some mechanism for doing this is attached to the stand of the pump.

106. The Condenser or Compressing Syringe.

This is an air-pump arranged to compress air into a vessel.

A piston P with a valve E opening into the barrel works in a barrel AB, Fig. 96 (a) The vessel into which the air is to be

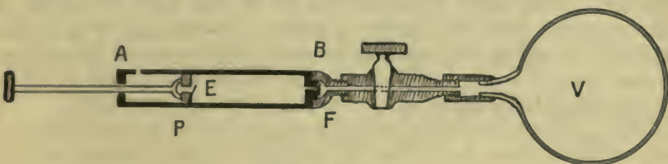


Fig. 96 (a).

compressed communicates with the end of the barrel through a tube. This tube is closed by a valve F opening outwards from the barrel. There is also usually a stopcock in the tube so that communication between the vessel and the air may be cut off at will. Let the piston be at the end of the barrel near the valve F. On withdrawing it the pressure in the barrel is reduced; the atmospheric pressure opens the valve E and the barrel is filled with air at atmospheric pressure. The piston is then depressed, the valve E is closed and F is opened;

hence all the air from the barrel is forced into the vessel; on again withdrawing the piston the process is repeated. At each downstroke a barrelful of air at atmospheric pressure is forced into the vessel.

A form of condensing pump used for the tyres of bicycle wheels is shewn in Fig. 96 (*b*). The piston rod DC is hollow

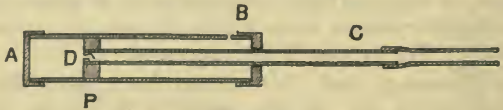


Fig. 96 (*b*).

and contains a valve at D. At C there is communication with the tyre. When the piston is pressed up against the end of the cylinder at B air can enter the cylinder behind the piston through the aperture B. The cylinder is then pushed forward and the piston moves past the aperture B, thus cutting off the cylinder from the atmosphere; as the piston is moved towards A the air in AD is compressed and forced through the valve D into the tyre.

We may find the density of the air, after any number of strokes, thus. Let V be the volume of the vessel, v that of the barrel. At each stroke a volume v at atmospheric pressure enters. Thus after n strokes the air in the receiver would, at atmospheric pressure, occupy a volume $V+nv$, and if ρ be the density of air at atmospheric pressure, the mass of air in the vessel is $\rho(V+nv)$. But its actual volume is V and, if ρ_n be its density, its mass is $\rho_n V$.

Thus $$\rho_n V = \rho(V+nv).$$

Hence $$\rho_n = \rho\left(1 + n\frac{v}{V}\right).$$

Again, by Boyle's Law the pressure of air is proportional to its density. Thus if p_n be the pressure after n strokes, π the initial pressure,

$$p_n = \pi\left(1 + n\frac{v}{V}\right).$$

Examples. (1) *If the volume of the vessel is ten times that of the barrel, how many strokes are required to double the pressure?*

We are to have $p_n = 2\pi$, also $v/V = 10$.

Thus $$2=1+\frac{n}{10}$$

or $$10+n=20,\ n=10.$$

Hence the pressure is doubled after 10 strokes.

(**2**) *When the piston of a condenser is pushed as far down as it will go a volume* v′ *is left beneath it. The valves will open when there is a difference of pressure* p *between the two sides. Shew that the pressure of the air inside the receiver can never be greater than* $(\pi - p)\,v/v' - p$, *where* π *is the atmospheric pressure.*

When the piston is drawn up there is a volume v of air below in the barrel; the pressure of this air is $\pi - p$[1].

As the piston is pushed down this air is compressed; let us suppose that its pressure at its greatest is just insufficient to open the lower valve, then when the stroke is complete its volume is v'; its pressure therefore is $(\pi - p)\,v/v'$. Since the valve just does not open the pressure inside is less than this by p. Hence the greatest pressure inside is

$$(\pi - p)\frac{v}{v'} - p.$$

107. The Diving-bell.

This is an apparatus for enabling a man to descend to a considerable depth under water, thus a sunken vessel could be examined, or the foundation of a pier laid or repaired, or work of other kinds carried out.

Take a beaker and immerse it mouth downwards in water: as the beaker is depressed the air it contains is compressed, but the water does not rise so as to fill the beaker completely, there is always air in the upper part of the beaker, a fly or small animal might live there for some time. The beaker is a diving-bell in miniature.

The bell, Fig. 97, consists of a large bell-shaped or cylindrical vessel closed at the top but open underneath. This can be lowered mouth downwards into the water, its weight is greater than the weight of water which would fill it, hence it sinks in the water. As it sinks the air it contains, like that in the beaker, is compressed, and the water rises in the bell but never fills it. The bell is so constructed that a person can stand within it and thus be lowered into the water without depriving him of air to breathe. The bell is usually filled with air by two tubes leading to the atmosphere above. Through

[1] If, as has been assumed in the text above, the valves open with no difference of pressure it would be π; it is less than this by the pressure required to open the valve.

one of these fresh air is forced into the bell, through the other foul air is withdrawn. The pressure in the bell depends on the depth to which it is sunk. The difficulty of working under very great pressure limits of course the depth at which it can be used.

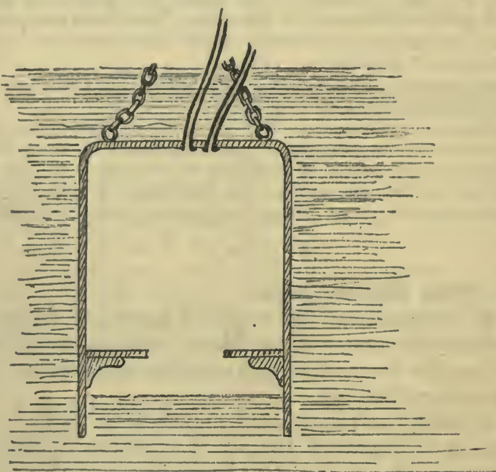


Fig. 97.

The tension on the chain supporting the bell is equal to the difference between the weight of the bell and the weight of water it displaces: as the bell sinks the water rises inside, the weight of water displaced therefore is reduced and the tension increased.

Examples. (1) *A conical wine-glass 4 inches in height is lowered mouth downwards into water until the level of the water inside is 34 feet below the surface; the height of the water-barometer being 34 feet, what is the height of that part of the cone which is occupied by air?*

The pressure in the wine-glass is doubled, thus the volume of air is half what it was.

But the volumes of two cones of the same angle are proportional to the cubes of their heights; hence, if z is the height of the conical volume of air in the bell, we have

$$\frac{z^3}{4^3}=\frac{1}{2}.$$

Therefore $$z=\frac{4}{\sqrt[3]{2}}=3\cdot175.$$

Hence the height required is 3·175 inches.

(2) *A cylindrical diving-bell* 6 *feet in height and* 5 *feet in diameter is lowered till its top is* 45 *feet below the surface. What volume of air at atmospheric pressure—that due to* 34 *feet of water—must be pumped in to fill the bell completely?*

[To solve this it is simplest to suppose the bell to be full and then to find what will be the volume at atmospheric pressure of the air actually in the bell.]

Let V be the volume of the bell.

When the bell is full the level of the water inside is $45+6$ or 51 feet below the surface. The pressure therefore is that due to a head of $(34+51)$ feet of water or $2\frac{1}{2}$ atmospheres. If the air then were at atmospheric pressure its volume would be $\frac{5}{2}$ of V. The volume of air added is at atmospheric pressure $\frac{3}{2}$ of V.

Now $$V=\frac{\pi}{4}\times 25\times 6 \text{ c. feet.}$$

Hence the volume required is $9\times 25\times \pi/4$ or 177·9 c. feet.

(3) *A piece of wood floats half immersed at the top of the water, how much of it will be immersed when floated in water within the bell mentioned in Example* 2?

Let $2v$ be the volume of the wood in cubic centimetres, σ the density of air at atmospheric pressure referred to water. Let $v+x$ be the volume of the wood in the air in the bell, $v-x$ then will be the volume in the water.

The density of the air in the bell is $\frac{5}{2}\sigma$.

The mass of air displaced in the first case is σv and of water it is v.

Since the wood floats the mass of the wood is equal to the mass of fluid displaced.

Therefore $\quad v+\sigma v=$ mass of wood in grammes.

Under the bell a mass $\frac{5}{2}\sigma(v+x)$ of air is displaced, the mass of water displaced is $v-x$.

Thus $\quad v-x+\frac{5}{2}\sigma(v+x)=$ mass of wood.

Hence, since the mass of the wood is unchanged,

$$v+\sigma v=v-x+\tfrac{5}{2}\sigma(v+x).$$

Therefore $$x(1-\tfrac{5}{2}\sigma)=\tfrac{3}{2}\sigma v.$$

Thus $$x/v=\tfrac{3}{2}\sigma/(1-\tfrac{5}{2}\sigma).$$

The fraction of the whole volume in the air is

$$\tfrac{1}{2}\left(\frac{v+x}{v}\right) \text{ or } \tfrac{1}{2}\left(1+\frac{x}{v}\right).$$

Since σ is approximately ·000129 this fraction is approximately $\frac{1}{2}(1\cdot000193)$.

*108. The Volumenometer.

This instrument, as its name indicates, is devised for the measurement of volume and is founded on an application of Boyle's Law. Let the top of the closed tube A in the apparatus shewn in Fig. 77 communicate with a vessel whose volume is to be found. Let V be the volume of this vessel, v the volume of unit length—1 centimetre—of the vertical tube. Let H be the height of the mercury barometer, a_1 the length of the vertical tube AB which is filled with air, h_1 the difference in level between the mercury columns in the two tubes.

The pressure of the air in the closed tube and vessel is measured by a head of mercury of height $H + h_1$, and the volume of the enclosed air is $V + a_1 v$.

Now raise the reservoir C; the mercury rises in AB. Let a_2 be the length of this tube now occupied by air and let h_2 be the difference in level between the two columns.

The air pressure is now measured by a head of mercury $H + h_2$, the volume of air enclosed is $V + a_2 v$.

But the mass and temperature of the air enclosed remain unchanged, hence by Boyle's Law the product of the pressure and volume is constant.

Therefore

$$(H + h_1)(V + a_1 v) = (H + h_2)(V + a_2 v).$$

Whence $$V(h_1 - h_2) = v\{a_2(H + h_2) - a_1(H + h_1)\},$$

and from this expression V can be found if v is determined by measuring the tube, and a_1, a_2, h_1, h_2 and H are observed.

In this way the volume of a closed vessel which can be connected to the volumenometer is found. To use the instrument to determine the volume of a solid, the solid is placed in the vessel whose volume V has previously been determined and the experiments are repeated in the same manner. Let V' be the volume of the solid and h_1', h_2', a_1', a_2' the new values of h_1, etc.

The volume of air in the vessel is now $V - V'$, for a volume V' of the interior is occupied by the solid.

Hence the above equation becomes

$$(V - V')(h_1' - h_2') = v\{a_2'(H + h_2') - a_1'(H + h_1')\}.$$

From this equation $V - V'$ can be found, but the value of V has already been determined, thus the value of V' is given.

A convenient form for the closed vessel is shewn in Fig. 98. It consists of a bulb of known volume V_0 opening into a funnel-shaped space. The upper edges of the funnel are ground flat and the whole can be closed in an air-tight manner by means of a ground glass plate and grease.

Fig. 98.

Two marks are made on the glass, the one at F between the bulb and the funnel, the other at G below the bulb; the volume of the funnel above F is V, the volume between F and G is V_0. To use the apparatus the glass plate is removed and the sliding reservoir adjusted so that the mercury fills the bulb and stands at the mark F. On replacing the plate and making the apparatus air-tight we have a volume V of air at atmospheric pressure H. When the reservoir is lowered the mercury sinks in the bulb. Lower it until the level of the mercury is at G, the mercury in the other tube will be below G, let h be the difference in level. The pressure of the air in the bulb and funnel is now $H - h$, and its volume is $V + V_0$.

Thus
$$(V + V_0)(H - h) = VH,$$
and
$$V = V_0 \frac{H - h}{h}.$$

Now repeat the experiment, placing the body of volume V' in the funnel, let h' be the observed difference of level.

Then
$$V - V' = V_0 \frac{H - h'}{h'}.$$

Hence
$$V' = V_0 \left\{ \frac{H - h}{h} - \frac{H - h'}{h'} \right\}$$
$$= V_0 \frac{H(h' - h)}{hh'}.$$

The method is chiefly useful in determining the volume of a light body of considerable size which cannot be easily weighed in water. By measuring with the balance the mass

of the body and dividing this by the volume the density can be found.

In this way the volume of a piece of pumice-stone, a quantity of glass-wool, or a number of feathers could be found.

EXAMPLES.

HYDROSTATIC MACHINES.

[For a Table of Specific Gravities see p. 15.]

1. The column of water in a siphon contains a small bubble of air, how is the working of the instrument affected?

2. How may a siphon be used to withdraw air from a vessel under water and open below?

3. A small hole is made in one leg of a siphon, how does this affect its working?

4. What height can (1) Mercury, (2) Alcohol be raised by a siphon?

5. A siphon tube leads from the bottom of a vessel, the top of the siphon being below that of the vessel. Water is allowed to drop into the vessel. Explain what happens and shew how this may be applied to explain the action of some forms of intermittent springs.

6. The piston of a common pump is 3 inches in diameter and the spout is 21 feet above the well. What is the force on the piston-rod when the pump is working?

7. In the same pump the length of stroke is 1 foot and the diameter of the pipe leading to the well is 1 inch, how many strokes must be made before the water will flow?

8. If in the same pump the distance from the fulcrum of the pump handle to the piston be 6 inches and from the fulcrum to the point at which the force is applied 4 feet, find the force required to work the pump.

9. What weight of water could be raised per hour by an engine of ·5 horse-power working the same pump?

10. A force-pump is employed to raise water a height of 60 feet, what work must be done to deliver 10 cubic feet of water per minute?

11. If the height of the cistern into which water is being pumped be 25 feet and the mechanical advantage of the handle 10, what force is needed to raise water with a pump whose piston is 4 inches in diameter?

The pump is worked at the rate of 10 strokes per minute and the handle moves through 3 feet at each stroke; find the work done and the weight of water raised in an hour.

12. Water is being lifted by a pump to a height of 30 feet, the diameter of the piston is 1 foot and the length of stroke is 3 feet; find the number of strokes per minute if 1500 lbs. are discharged in that time.

13. A force-pump raises water a height of 20 feet and forces it a further height of 60. The diameter of the piston is 6 inches; find the force on the piston rod during the back and forward stroke.

14. The area of the piston of a force-pump is 1 square foot and the length of stroke 5 feet. Find the work done during one down stroke if water is being raised to a height of 50 feet above the pump.

15. How high could a liquid of specific gravity 1·34 be raised by a suction-pump?

16. If the volume of the receiver of an air-pump be three times that of the barrel, calculate in terms of the initial pressure the pressure after 5, 10, and 15 strokes.

17. If with the same pump the gauge originally stood at 30 inches, determine the number of strokes after which it will stand at 25 inches.

18. Find the number of strokes which with the same pump will reduce the pressure to 1 per cent. of its initial value.

19. If the volume of the receiver of a Smeaton's pump be 3 times that of the barrel, at what point of the third stroke will the valve at the top of the barrel open?

20. The range of a Smeaton's pump is 5 inches and the piston at the top and the bottom of its stroke is $\frac{1}{3}$ inch from the ends of the cylinder, determine the minimum density of the air in the receiver neglecting the difference of pressure required to open the valves.

21. Find the ratio of the volume of the receiver to that of the barrel if at the end of the fourth stroke the density of the air is $\frac{256}{625}$ of its original density.

22. Describe some form of air-pump. If the size of the receiver of an air-pump be 1 cubic foot and that of the barrel of the pump 24 cubic inches, how many strokes are required to reduce the pressure of the air to one-tenth of the atmospheric pressure?

23. The volume of the receiver of an air-pump is 4 times that of the barrel. Shew that after 5 strokes the air will be reduced to less than one-third of its initial density.

24. The volume of the receiver of an air-pump is 500 c. cm., and that of the barrel 75 c. cm., find after how many strokes the pressure will be reduced to less than a half of its original value.

25. The barrel of a Smeaton's air-pump is of the same capacity as the receiver and connecting tube. Supposing that the valve at the bottom of the cylinder is the first to cease working, and that there is no leakage, shew that the most complete exhaustion the air-pump can give will be accomplished at the end of 8 strokes. The area of the valve is supposed to be $\frac{1}{1000}$ of a square inch, and its weight $\frac{11}{1200}$ths of an ounce, the atmospheric pressure being just under 2112 lbs. weight per square foot.

26. In a Geissler pump the fixed reservoir contains 1000 c. cm. Find the density of the air remaining in a bulb 100 c. cm. in volume after 3 strokes of the pump.

27. Assuming the volume of the tube of a bicycle tyre to be 100 cubic inches and the barrel of a pump used to fill it to be 10 cubic inches, find the number of strokes required to produce double the atmospheric pressure in the tyre.

28. Determine the volume of the barrel of a condenser if one stroke is sufficient to produce a pressure of 5 atmospheres in a tube $\frac{1}{2}$ an inch in diameter and 50 feet long.

29. A speaking tube, of 1 square inch section, is found to be blocked somewhere. A condensing pump, the capacity of whose barrel is 60 cubic inches, is attached to the mouth of the speaking tube, and after 40 strokes the pressure of air in the tube is found to be 3 atmospheres. Shew that the block is 100 feet from the mouth of the tube.

30. A cylindrical diving-bell 9 feet long is sunk in water to such a depth that the water rises $3\frac{1}{2}$ feet in the bell. At what depth is the surface of the water inside the bell if the water barometer stands at 34 feet?

31. A cylindrical bell 10 feet in height is sunk under the sea until the water rises halfway up the bell; find the depth of the top of the bell, taking the height of the water barometer as 33 feet.

32. The same bell is sunk in fresh water and the water rises 2 feet in the bell, find the depth to which it is sunk.

33. What volume of air at atmospheric pressure must be pumped in to fill the bell in each of these two cases?

34. A diving-bell is let down into water so that the level of the water in the bell is 33 feet below the surface of the water. If the bell is cylindrical and no air is pumped into it whilst it goes down, how high will the water have risen in the bell itself?

35. The capacity of a diving-bell is 100 cubic feet. What volume will the air, which fills it at a depth of 60 feet, occupy when raised to the surface, the height of the water barometer being taken as 30 feet?

36. A cylindrical diving-bell is lowered to such a depth that the confined air occupies two-thirds of the interior. Half as much air again is pumped into the bell. How much further may the bell descend before it becomes half full of water?

37. The height of a cylindrical bell is a feet at the surface, a mercury barometer reads h feet, when the bell is sunk it reads h' feet. If ρ be the specific gravity of mercury, find the depth to which the bell has sunk.

38. The diameters of the pistons of a Bramah press are 1 inch and 20 inches respectively, find the weight which can be raised if a force of 2·5 tons weight be applied. At each stroke the small piston moves over 10 inches, find the number of strokes required to raise the weight 25 feet.

39. A lift constructed to carry 10,000 lbs. weight is supplied with water from a height of 150 feet. Find the diameter of the piston.

40. If the sections of the cylinders of a Bramah press be 18 square inches and 1 square foot respectively, what pressure must be applied to the smaller cylinder to produce a pressure of two tons upon the larger?

41. The diameter of the large piston of a press is 10 times that of the small one, and at each stroke the small piston moves through 5 inches. What weight can be raised by a force equal to the weight of 28 lbs. applied to the smaller piston, and how many strokes are required to raise the weight 1 foot?

42. The ram of a press is 20 inches in diameter, and it is required to lift 100 tons, what size should you make the plunger of the pump, if the mechanical advantage of the handle be 10 and the force on the man's hand working the pump 10 lbs. wt.?

43. If the ram of an hydraulic accumulator be 6 inches in diameter, what load is required to produce a pressure of 500 lbs. on the square inch? To what head of water does this correspond?

44. The mechanical advantage of the arm of a safety-valve is 5, and the diameter of the steam valve is 1 inch. If the load at the end of the arms be 30 lbs. weight, find the steam pressure.

45. In a volumenometer the volume of the space into which the body to be measured is put is 100 c.cm., that of the measured space below is 50 c.cm. When a body 25 grammes in mass is inserted and the mercury is brought to the lower mark the pressure is found to be 38 cm.; find the density of the body.

46. In another experiment with the same instrument when the mercury is brought to the lower mark the pressure is 46 cm., find the volume of the body introduced.

47. A horizontal waterpipe is connected with a reservoir; the level of water in the reservoir is 300 feet above the pipe. The pipe whose internal diameter is 1 inch, is cut and stopped by a cork. Find the force exerted by the water on the cork.

48. A safety-valve is kept in position by a horizontal lever 12 inches long having its fulcrum at one end and a weight of 5 lbs. on the other. The valve itself is one square inch in section and its centre is at a distance of one inch from the fulcrum. What is the greatest pressure that the steam in the boiler can have?

49. The safety-valve of a steam-engine is one square inch in section, and its centre is placed 4 inches from the end of a 20 inch lever, from the other end of which a weight of 27 lbs. is suspended. If the atmospheric pressure be 15 lbs. per square inch, find the maximum pressure of the steam in the boiler, when it begins to escape by the valve.

50. The weight of a cubic foot of water being 62 lbs., what volume of water will an engine of 12 horse-power raise by pumping for 1 hour from a well 20 feet deep? If the loss of work from frictional causes be equivalent to 1000 foot lbs. per second, calculate by how much the volume raised will be reduced.

EXAMINATION PAPERS.

I.

1. Distinguish between a solid and a fluid, giving examples and shewing how the substances mentioned conform to the definition.

2. What is meant by viscosity and plasticity? Why is pitch a viscous fluid while wax is a plastic solid? What do you understand by elasticity?

3. Define the pressure at a point in a fluid, and shew that the pressure at any point is the same in all directions. Shew also that in a fluid under gravity the pressure is the same at any two points at the same depth.

4. Shew that if p_1, p_2 be the pressures at two points in a fluid of density ρ and h the vertical distance between the points, $p_2 = p_1 + h\rho g$.

5. Shew that the resultant thrust on an area immersed in a fluid is equal to the weight of a column of fluid whose base is the area, and whose height is the depth of the centre of gravity of the area.

6. Explain how the pressure of the air is measured by the barometer.

7. Explain the action of the siphon.

II.

1. Describe the construction and mode of action of the barometer. When a barometer tube is inclined the top of the column remains at the same vertical height above the mercury in the dish. Why is this? Does the height of the barometer depend on the area of the cross section of the tube?

2. Explain the mode of action of the common pump, the force-pump, and the air-pump.

3. Shew both from theory and experiment that the resultant force on a body immersed in a fluid is equal to the weight of fluid displaced. Deduce hence the laws of floating bodies.

4. Define specific gravity, distinguishing carefully between it and density.

5. Describe the use of the hydrostatic balance, and shew how to employ it to find the specific gravity of a solid lighter than water.

6. Shew how to find the specific gravity of a liquid (*a*) by the use of the specific gravity bottle, (*b*) by the use of Nicholson's hydrometer.

7. A long U-tube contains two liquids which do not mix. Shew how to compare the densities of the two by measuring the heights of the respective columns above the common surface.

ANSWERS TO EXAMPLES IN DYNAMICS.

CHAPTER I. (Page 18.)

1. (1) 35·558 cms.; (2) 182·87 cms.; (3) 152·39 cms.; (4) 20115·82 cms.
2. (1) 1·60927; (2) 6437·08; (3) ·5486; (4) ·001; (5) ·000025.
3. (1) 191·58; (2) 5747·38; (3) 4046,153,000; (4) 471,428·6.
4. (1) 2919·52; (2) 2627570; (3) 1302266×10^{12}; (4) 1768×10^{7}.
5. 56π sq. in. = 176 sq. in.
6. $\frac{1}{4\pi}$ sq. miles = 246,400 sq. yds.
7. 42 ft.
8. The circle.
9. $25\sqrt{3}$ sq. ft. = 43·3 sq. ft.
10. $\frac{20\sqrt{2}}{\sqrt{3}}$ cm. and $\frac{20}{\sqrt{3}}$ cm.
11. 800 sq. cm.
12. 2·10 mm., 55·44 sq. mm.
13. ·428 mm.
14. 112·62 grms.
15. 76·37 lb. per c. foot.
16. 1·19 grms. per c. cm.
17. 1·22 grms. per c. cm.
18. 18·5 grms. per c. cm.
19. 3437 grains per c. inch.
20. The density of the sphere = the density of the cylinder × 7·80.

CHAPTER II. (Page 53.)

1. (1) $14\frac{2}{3}$; (2) 30; (3) $981,333,333\frac{1}{3}$; (4) $1527\frac{1}{3}$.
2. (1) $\frac{a}{b}$; (2) $62\frac{6}{7}$; (3) $\frac{19}{150}$ = ·1267; (4) 2·032.
3. 1117·545.
4. (1) 2640; (2) 3801600; (3) 1387584000.
5. (1) $8\frac{8}{11}$ ft. per sec.; (2) $5\frac{5}{7}$ miles per hour.
6. 4 miles from *A*'s starting-place.
   A's speed is 4·8 miles per hour.
   B's ,, 2·4 ,, ,,
7. 5 miles.
8. 2750 ft.
9. $\frac{5}{8}$ ft. per second.
10. 248 to 256 approximately.
11. (1) 5; (2) 10; (3) 19·21; (4) 5.
12. (1) $\sqrt{37}$ = 6·083; (2) 12·96; (3) 2·91; (4) 2·65.

13. 5 miles per hour.
14. 16·1 ft. per second.
15. $50\frac{3}{8}$ minutes from start. $2\frac{3}{8}$ miles.
18. (1) $\sqrt{2}$; (2) $2\sqrt{2}$; (3) $2\sqrt{5}=4{\cdot}47$; (4) $3\sqrt{2}=4{\cdot}24$.
19. (1) 0; (2) $\sqrt{3}$; (3) $\sqrt{19}=4{\cdot}36$; (4) $3\sqrt{3}$.
20. (1) $500\sqrt{3}$ horizontally, 500 vertically;
    $500\sqrt{2}$ „ $500\sqrt{2}$ „
    500 „ $500\sqrt{3}$ „

    (2) $\dfrac{55}{\sqrt{3}}=31{\cdot}8$ ft. per sec. horizontally, $18\frac{1}{3}$ ft. per sec. vertically.
21. 577 ft. per sec.
22. $250\sqrt{3}$ ft. per sec. $=433$ ft. per sec.
23. $5\sqrt{2}$ miles per hour in the north-west direction.
24. BC.
25. (a) $35\sqrt{2}$ cms. per sec. to the north-west; (b) 6.
26. $30\sqrt{3}$ miles per hour.
27. 900 ft.
28. The velocities during each of the 4 minutes are 2, 6, 10 and 14 ft. per second respectively.
29. $\dfrac{t^2}{2}$ ft.
30. The angle between the two components is $\cos^{-1}=\frac{1}{15}$.
32. $\frac{1}{2}AB$.
33. $nu\tau+v\tau^2\dfrac{n(n-1)}{2}$.
34. 73·3 to 55.

CHAPTER III. (Page 73.)

1. $\frac{15}{2}g$ and $\frac{19}{2}g$.
2. $\frac{3}{2}g$.
3. 15625 ft. 62·5 sec.
4. (1) 12·5; (ii) 26·25.
5. 15·625.
6. Change in velocity $=5\sqrt{2}$. Acceleration $=\dfrac{1}{\sqrt{2}}$.
7. 1150 ft.
8. $a=\frac{44}{135}$ $(=\cdot 326)$ ft. per sec. per sec.
9. $u=160\sqrt{13}$ $(=577)$ ft. per sec. Height $=5200$ ft.
10. 420000 ft. 1300 ft. per sec.
11. 36 sec.
12. (a) 3 sec. $13\frac{1}{2}g$. (b) $1\frac{1}{2}$ sec. $3\frac{3}{8}g$.
13. $25\left(1-\dfrac{1}{\sqrt{2}}\right)=7{\cdot}3$ sec. $\dfrac{25}{\sqrt{2}}=17{\cdot}7$.
14. $\dfrac{u}{a}$.
15. Half the height. $\sqrt{\text{height}\times g}$.
16. 3 sec.
17. 272 ft. per sec.
18. Initial velocity $=0$.
19. 976·56 ft.
20. 4 sec.
21. $833\frac{1}{3}$ cm. per sec. $3\frac{17}{36}$ cm. per sec. per sec.
22. 4587·2 metres. After 61·16 secs.
23. 88·6 metres per sec.
24. 38624.
25. 10·2 sec.
26. 144 ft.

27. $\frac{11}{16}$ ft. per sec. per sec. 28. $12\frac{2}{3}$ ft. per sec. No.
29. 16 ft. per sec. $a=\frac{2}{15}$ ft. per sec. per sec.
30. $-\frac{121}{2160}$ $(=-\cdot056)$ ft. per sec. per sec. 31. $13\frac{1}{3}$ secs.
34. 25 ft. 40 ft. per sec. 36. $61\frac{5}{11}$ ft. per sec. $276\frac{9}{11}$ ft.
37. velocity at end of 5 sec. = 320 ft. per sec.
initial velocity = 20 ft. per sec.
38. 120 ft. 39. 120 ft.
40. At an angle $\cos^{-1}\frac{1}{3}$ with the direction of motion of the train.
41. $6\frac{2}{3}$ miles per hour. $35\frac{1}{5}$ ft. per sec.
1 mile in $6\frac{1}{5}$ min. = 14·2 ft. per sec. 42. tangent $=\frac{4}{3}$.
43. velocity $=\frac{3\pi}{10}$ ft. per sec.; angular velocity $=\frac{\pi}{10}=(18^\circ)$ per sec.
44. mean acceleration $=\frac{v-u}{t}$. 46. 100 ft. 80 ft. per sec.
47. $32\sqrt{5}$ ft. per sec. $\sqrt{5}$ secs.
48. 14 ft. per sec. 24 ft. per sec. per sec. 49. 159 ft. approx.
50. 3888000 miles per hr. per hr.

CHAPTER VII. (Page 142.)

1. (i) 100,000 units of momentum;
(ii) $20\frac{5}{6}$; (iii) 2,682,240; (iv) 9,900,000.
2. momentum of second mass = momentum of first mass × 231·5.
3. $27\frac{7}{9}$ cm. per sec. per sec. $1,666,666\frac{2}{3}$ dynes.
4. impressed force on second mass
= impressed force on first mass × ·2572.
5. $469\cdot5\pi\times10^{31}$ c.g.s. units of momentum. 6. 45·4 cm.
7. impulse $=30\sqrt{5}\times m$, where m = mass of cricket-ball;
average force $=1500\sqrt{5}\times m$ poundals.
8. 2215 cm. per sec. 9. 256 lbs. 8192 poundals. 11. $62\frac{1}{2}$ poundals.
12. $363\frac{12}{37}$ cm. per sec. $181\frac{33}{37}$ cm. $21822\frac{2}{3}$ cm. 13. 24 ft. per sec.
14. $10\frac{2}{3}$ pounds. 15. $21\frac{9}{11}$ pounds. 16. 60,500 ft. $=11\frac{11}{24}$ miles.
17. $-1\frac{7}{8}$ ft. per sec. per sec. $\frac{1}{2160}$ of the weight.
20. forces are equal. 21. 200 poundals. 22. $2133\frac{1}{3}$ poundals.
23. a dyne. 25. $17\frac{1}{7}$ poundals. $28\frac{1}{8}$ ft. 26. $5\sqrt{30}$ mm. = 27·4 mm.
27. $\frac{80\sqrt{5}}{3}=59\cdot6$ ft. per sec. 28. 16 ft. per sec.
29. 255·7 cm. per sec. 30. 38080 poundals.
31. $2\frac{2}{3}$ lb. weight. $10\frac{2}{3}$ ft. per sec. per sec.

32. 38·8 ft. per sec. per sec. **33.** 14·85 cm. per sec. per sec.

34. $5\frac{1}{3}$ ft. per sec. per sec. 16 ft. per sec. 24 ft.

35. 8·83 cm. per sec. per sec.

36. (1) $\frac{32h}{l}$ ft. per sec. per sec. (2) $\frac{l}{4\sqrt{h}}$ secs. (3) 16 ft. per sec.

37. (1) $\frac{g \, . \, h}{l}$. (2) $\frac{l\sqrt{2}}{\sqrt{h \, . \, g}}$ secs. (3) 28 ft. per sec.

41. $18\frac{1}{3}$. **42.** 2. **43.** $\left(\frac{n'-n}{n'+1}\right)g$.

44. (1) $2mu \sin \frac{1}{2}a$. (2) $\frac{P}{2m} - \mu g$. (3) $\frac{P}{2} + \mu mg$.

45. 313 cm. per sec. **46.** (1) 30°. (2) 56·6 cm. per sec.

47. (1) $2\frac{2}{3}$ ft. per sec. per sec. (2) 24 F.P.S. units of momentum.

48. (1) $5\frac{1}{3}$ ft. per sec. per sec. (2) $106\frac{2}{3}$ poundals. (3) $66\frac{2}{3}$ ft.

49. (1) 3 tons' wt. (2) 525·2 ft. **50.** 1 kilo. wt.

51. 20 stone wt. **52.** $288\sqrt{2}$ ft.

54. 45 poundals. **55.** 125 ft.

56. (1) 2 ft. per sec. per sec. (2) 150 poundals and 136 poundals.

57. (1) $\frac{2 \, . \, m' \, . \, m \, . \, g}{m'+m}$. (2) $1\frac{1}{2}$ secs.

CHAPTER VIII. (Page 184.)

3. $\frac{10^4}{g^2} = \cdot 0104$ secs.

4. (1) momentum of bullet = 37·5 units;
energy ,, = 22,500 foot-poundals.
(2) momentum of large mass = 560 units;
energy ,, ,, = 140 foot-poundals;
force to stop bullet in $\frac{1}{10}$ sec. = 375 poundals;
,, ,, large mass ,, ,, = 5600 ,,
work done by bullet = 22,500 foot-poundals;
,, ,, large mass = 140 ,,

6. $150g \, . \sqrt{2}$ ergs. **8.** 9 cm. per sec.; before impact = 3840 ergs.
after ,, = 3240 ,,

9. (1) 1792 ft. per sec. (2) $16\sqrt{7} = 42 \cdot 4$ ft. per sec.

10. (1) 19,200 F.P.S. units of momentum.
(2) $1645714\frac{2}{7}$ ft.-poundals. (3) $1645714\frac{2}{7}$ ft.-poundals.

11. (1) 31,250 ft.-poundals. (2) $10,416\frac{2}{3}$ poundals. **12.** 149 oz. 2·5 ft.-tons.

13. $\frac{4}{15}$ horse-power. **14.** 5 ft.-pounds. **15.** 5·7 horse-power.

16. When the pendulum has fallen through half the vertical height of its swing.

17. 5,040,000 ft.-poundals. **18.** 3520 ft.-pounds.

19. (1) 20 ft. per sec. (2) 1 to 100.

20. $17\frac{1}{7}$ ft. per sec. **21.** 1·59 H.P. **22.** $74\frac{2}{3}$ H.P.

23. $2199\frac{3}{11}$ H.P. **25.** 4 inches. **28.** $7812\frac{1}{2}$ poundals.

29. 31,500 poundals. **30.** 2685×10^{10} ergs. **31.** $68\frac{3}{4}$ c. ft.

CHAPTER IX. (Page 208.)

1. $40\sqrt{2}$ yds. **2.** 14·06 ft. **4.** (1) 108 ft. (2) 3 secs.

5. (1) 1000 ft. per sec. (2) 144 ft. **6.** (1) 13 secs. (2) $104g$ ft.

7. $\dfrac{55 \sin 2\alpha}{12}$ miles. **9.** 100 ft. per sec. **10.** 70 ft. per sec.

12. (1) 22,050 ft. (2) 553·48 ft. per sec. **14.** $\dfrac{-V^2 \cos 2\alpha}{2g}$.

CHAPTER X. (Page 221.)

1. $\frac{2}{3}$ of that of the impinging ball. **2.** 2 to 5. $\frac{3}{4}$.

4. 15 ft. per sec. in opposite directions.

5. Their masses are equal and e is unity. **6.** ·0878 ft.

7. $\tan^{-1}\frac{1}{3}$. **8.** $v=\frac{1}{2}u$; $\beta=90^\circ$. $v'=\dfrac{\sqrt{3}}{2}u$; $\beta'=0^\circ$.

9. $\cos\alpha=\dfrac{v}{2u}$. **11.** $\dfrac{1-e^2}{4}.u$.

CHAPTER XI. (Page 236.)

2. 1·65 revolutions per second. **3.** lengthened by ·08 inches.

5. (1) In first position pressure on support $= 800\sqrt{\frac{2}{3}}$ grms. wt.,

tension of string $= 200\sqrt{\frac{2}{3}}$,,

weight of bob (1000 grms. wt.).

(2) In second position pressure on support $250\sqrt{15}$ grms. wt.

(3) In third position pressure on support $\left(1+\dfrac{1}{2\sqrt{15}}\right)$ kilos. wt.

7. Intensity of gravity at B = intensity at $A \times 1{\cdot}00023$.

14. $\cdot 804$ ft.-pounds. 15. 24 ft. per sec. 16. 80854 ergs.

23. $40\sqrt{6}$ ft. per sec. 24. 9 to 8. 25. 1.

27. $28\sqrt{2}$ ft. per sec. $36\frac{3}{4}$ ft. 28. $\frac{3}{4}$ of the way across

29. 3 times as far as it fell in the first case.

30. $20\frac{1}{4}$ ft. After $\frac{3}{4}$ sec. and $1\frac{1}{2}$ secs. 31. $31{\cdot}6$ ft. per sec.

33. (1) $13\frac{1}{6}$ ft. per sec.

(2) $\text{kinetic energy of gun} = \dfrac{\text{kinetic energy of projectile}}{200}$.

34. (1) $4\frac{4}{7}$ ft. per sec. per sec. (2) $109\frac{5}{7}$ poundals.

35. (1) $8\frac{1}{3}$ miles per hour. (2) $933{,}333\frac{1}{3}$ ft.-poundals.

36. (1) $\frac{32}{33}$ ft. per sec. per sec. (2) 33 poundals.

MISCELLANEOUS EXAMPLES. (Page 238.)

3. Increased 12,960 times. 4. $\dfrac{1}{\sqrt{g}}$ secs.

6. $1\frac{19}{72}$ lb.-wt. 7. $\frac{14}{15}$.

8. 45° to the direction in which the ball is coming. 9. 400 ft.

10. 7 miles per hour. 12. 1108 ft. per sec.

13. $5\sqrt{13}$ miles per hour. 14. Increased 3600 times.

15. $6\frac{3}{4}$ miles. 16. $\frac{11}{540}$ ft. per sec. per sec.

17. $15\sqrt{3}$ miles per hour. 18. 5 ft. per sec. per sec.

19. $\frac{6}{7}$ secs. 21. $\frac{\pi}{2} + \sin^{-1}\frac{1}{5}$, with the motion of the train.

22. $4{\cdot}65$ secs. 37. (1) 448 ft.-poundals. (2) 18 ft.

40. $195\frac{5}{16}$ ft.-lb. 41. 22500 ft.-lb.

42. (1) 22400000 ft.-lb. (2) $678{\cdot}8$ H.P. 43. $\frac{5}{12}$.

44. 9856 ft.-lb. 47. $\frac{25}{64}$ ft. per sec. 48. $3\frac{9}{32}$ pounds.

49. (1) 19200 F.P.S. units of momentum. (2) $1648571\frac{3}{7}$ ft.-poundals.
(3) $1648571\frac{3}{7}$ ft.-poundals.

50. (1) $\frac{3}{50}$ ft. per sec. per sec. (2) $147\frac{3}{11}$ miles per hour.

51. $8\sqrt{6}$ ft. per sec. 52. 11 ft.-poundals. 54. $38554687{\cdot}5$ ft.-lb.

ANSWERS TO EXAMPLES IN STATICS.

CHAPTER I. (Page 31.)

1. $5\sqrt{13}$ lb.-wt. **2.** $13P$. **4.** 10 lb. and 26 lb.

5. 5 lb. and 13 lb. **13.** $5\sqrt{3}$ lb. each.

14. Place the forces parallel to the sides of a right-angled triangle whose sides are 3, 4 and 5.

16. Weight of bob = 1 kilogramme.
Tension of horizontal string = ·258 kilogramme.
" pendulum " = 1·033 "

17. $P(2+\sqrt{3})$. **21.** $\frac{\sqrt{7}}{2}$ times the side of the triangle.

26. A force equal to the given forces bisecting the angle between them

29. (i) $\sqrt{P^2+Q^2+\sqrt{2}\,.\,P\,.\,Q}$. (ii) 6·48 lb.-wt.

32. $5\sqrt{7}$ lb.-wt. **33.** 699·5 dynes.

34. [illegible] lb.-wt. and [illegible] lb.-wt.

38. $5\sqrt{2}$ to the North-west, 5 to the South.

41. $\sin^{-1}(-\frac{12}{13})$ with the force 5. $\sin^{-1}\frac{5}{13}$ with the force 12.

44. 27·7 oz. **46.** $\sqrt{2}\,.\,P$. **47.** $\sqrt{2}\,.\,P$.

48. 39 lb. **49.** 6 lb. and 8 lb.-wt.

CHAPTER II. (Page 57.)

2. 21 ft. 3 in. **3.** 3 lb.-wt. **5.** 5 lb.

6. 17 lb. **7.** $5\frac{1}{4}$ lb.-wt. **8.** 35 lb. and 40 lb.

9. $AC = 2$ in. $BC = 56$ in. **10.** [illegible] and [illegible].

12. 100 lb. **13.** ·131 ft. from the axle. **15.** $1\frac{1}{2}$ ft. from the man.

16. $2\sqrt{2} \,.\, P$ at an angle of 45° with the force $4P$.

17. In the line joining the middle points of the two sides, and at a distance of $5\frac{5}{8}$ in. from the side which supports the two 5 lb.-wt.

18. $\frac{2}{5}$ lb.-wt.

CHAPTER IV. (Page 83.)

1. $\frac{W}{2}$; $\frac{\sqrt{3} \,.\, W}{2}$. **3.** 50 lb. 4 ft. from the thicker end.

4. Pressure on shorter end $= \frac{6}{7}$ of the man's weight.
   ,, longer ,, $= \frac{1}{7}$,,

6. P, $\frac{1}{3}AC$ from C. **10.** $\frac{1}{2}W \tan \frac{1}{2}a$.

12. The radius to P bisects the line AB. **13.** 20 lb.

15. The end. **18.** 3 to 1.

CHAPTER V. (Page 110.)

1. 350 lb.

3. From any point on the rim within a distance of $\frac{1}{24}$ of the whole circumference from any of the 4 legs.

4. At a distance of $\frac{1}{6}$ of the side of the square from the centre of the square; the straight line joining the centre of gravity with the centre of the square is parallel to a side of the square.

5. At a point on the diameter drawn from the middle one of the 3 particles mentioned and at a distance of $·65d$ from that point where d is the length of the diameter.

6. In the perpendicular drawn from the angle which is adjacent to the 2 bisected sides, and at a distance $\frac{a}{6\sqrt{3}}$ below the C. G. of the whole triangle.

7. In the diameter of the rectangle parallel to the side a and at a distance of $\frac{1}{6}a$ from G.

8. At the point G in the straight line AC where $AG = \frac{5}{12} \,.\, AC$.

9. In the diameter drawn from that angular point on which no weight is placed and at a distance of $\frac{9}{16}$ of the diameter from that point.

11. In the straight line drawn parallel to BC from the middle point of AB and at a distance of $\frac{26}{45}$ of the side of the square from this point.

12. Half the weight of the whole table.

13. In the diameter drawn from the point at which the two circles touch one another and at a distance of $\frac{r^2}{R+r}$ from the centre of the larger circle.

14. In the diagonal drawn from the angle enclosed between the two bisected sides and at a distance of $\frac{23}{42}$ of the diagonal from this point.

18. 50 lb. 6 ft. from thinner end. 19. $\frac{\sqrt{3}}{2} W$. $\frac{1}{2} W$.

20. $\frac{2}{1+4\sqrt{3}}$ lb. 2 cwt., 3 cwt., 0.

22. In the straight line joining the two centres, and at a distance of 1 ft. $8\frac{1}{3}$ in. from the centre of the hole.

23. The centre of the circular hole must be 16 in. from that of the disc.

24. $1\frac{1}{3}$ in. 25. $5\frac{13}{47}$ in. from the thicker end.

27. On the line joining the angle removed to the c. of G. of the whole and $\frac{1}{6}$ of this distance from the angle removed.

28. $\frac{5}{16}$ of total length from the heavier end. 29. 5 ft. from the end.

30. In the diagonal drawn from the angle enclosed between the two bisected sides and at a distance of $\frac{23 . \sqrt{2}}{42}$ of the side of the square from this point.

CHAPTER VI. (Page 160.)

1. Left arm = right arm $\times$ 1·018.

2. The fulcrum is at C, that is, $13\frac{1}{3}$ ft. from A.

3. $\frac{3}{8}$ of distance of centre of gravity from the end at which the weight is hung.

12. True weight = 20·494 lb.

13. $7\frac{1}{2}$ pence. 15. $\frac{1}{2} W$. 16. $\sqrt{3} W$. 17. 12 ft.

18. $\frac{W . \sin \alpha}{\cos \theta}$. $W (\cos \alpha \pm \sin \alpha \tan \theta)$ depending on direction of force. Vertically up.

19. $\frac{W . \sin \alpha}{\cos \beta}$. Nothing. 20. $\frac{1}{\sqrt{2}}$ ton. 21. 20 lb.

22. $\frac{1}{\sqrt{3}}$ ton. 23. $\frac{W}{\sqrt{8}}$. 24. W, W.

25. $1 : \frac{1}{\sqrt{3}}$. When the force acts parallel to the plane.

26. 5 lb.-wt. **27.** 5 lb. **28.** $3\frac{1}{3}$ lb.
29. (1) 1 : 16 ; (2) 9 in. **30.** 7 lb. **31.** 16 lb. **32.** $9\frac{1}{3}$ lb.
33. $5\frac{1}{4}$ lb. **34.** $4\frac{1}{4}$ lb.
35. From a point distant $\frac{2}{7}$ of the whole length of the bar from the end to which the string attached to the fixed pulley is connected.
38. $\frac{W}{P} = \frac{\text{radius of wheel}}{\text{radius of axle}}$. **39.** 12 lb.
40. (1) 45·72 grams; (2) 1·002 to 1. **41.** $21\sqrt{2}$ lb. 21 lb.
42. 2 lb. **43.** 16 ft. **44.** (1) $\frac{W}{15}$. (2) $4\frac{1}{2}$ lb.
45. (1) $\frac{W}{16}$. (2) 5 lb. **46.** 7 pulleys. 14 lb. **47.** $\frac{13}{126}$ lb.

CHAPTER VII. (Page 174.)

1. $\sqrt{\dfrac{2s}{g \sin \alpha - \mu g \cos \alpha}}$. **3.** $\tan^{-1} \frac{8}{3}$, $\tan^{-1} \frac{3}{8}$. **6.** $2 \sin \alpha$.
9. $\sqrt{2}$ lb. ; $\tan^{-1} \dfrac{1}{\sqrt{3}}$ to the horizon ; $\dfrac{\sqrt{3}}{2}$ times the weight.
11. With the largest face on the plane and the length up the plane. No.
12. $30°$; $\dfrac{1}{\sqrt{3}}$. **13.** $\dfrac{1}{\sqrt{3}}$; $\dfrac{50}{\sqrt{3}}$ lb.
14. $2\frac{1}{4}$ cwt. **17.** $\frac{1}{2}$ cwt.; 0.

ANSWERS TO EXAMPLES IN HYDROSTATICS.

CHAPTER I. (Page 18.)

5. Density, 112 lb. per cubic foot. Specific Gravity, 1·792.
6. 112·58 grms.
7. 76·39 lb. per c. ft.
8. 1·1937 grms. per c. cm.
9. 1·2237 grms. per c. cm.
10. 18·476 grms. per c. cm.
11. 3437 grains per c. inch.
12. $\dfrac{\text{Density of sphere}}{\text{Density of cylinder}} = 7·803.$
13. 44·68 lb.
14. 199,200 tons.
15. 53 litres.
16. 72·4 litres.
17. (i) ·975. (ii) 1·027.
18. 7·09.
19. ·00459 sq. cm.
20. 2·82 c. inches.
21. $\dfrac{\text{Weight of glycerine}}{\text{Weight of water}} = ·713.$
22. 2·98 c. cm.

CHAPTER III. (Page 76.)

1. Pressure in *water* at depth of

(1) 25 cm. is 1025 grms.-wt. per sq. cm.
(2) 1 metre is 1100 ,, ,, ,,
(3) 1 mile is 161,900 ,, ,, ,,
(4) 5 kilometres is 501,000 ,, ,,

Pressure in *mercury* at depth of

(1) 1 cm. is 1013·6 grms.-wt. per sq. cm.
(2) 1 metre is 2360 ,, ,, ,,
(3) 25 metres is 35,000 ,, ,,
(4) 1 kilometre is 1,361,000 ,,

2. 44·91 ft.
3. 40·66 inches.
4. ·795 p, 1·28 p, p, where p is the pressure in the water.

5. Head of water $=33 \cdot 4$ ft. Head of mercury $=29 \cdot 48$ inches.
6. $130 \cdot 2$ lb.-wt. per sq. inch.
7. Heads of water, 10 metres, $829 \cdot 4$ inches, $1019 \cdot 4$ cm. Heads of mercury, $73 \cdot 53$ cm., $60 \cdot 98$ inches, $74 \cdot 95$ cm.
8. 68,400 poundals.
9. $(\Pi + 160 \cdot 3)$ lb.-wt. per sq. in. where Π = pressure of the atmosphere in lb.-wt. per sq. inch.
10. $1584 \cdot 5$ lb.-wt. per sq. ft.
11. 39270 lb.-wt.
12. Pressure in liquid $=178 \cdot 25$ lb.-wt. per sq. inch. Force exerted by piston $=14,000$ lb.-wt.
13. (1) 25,920 lb.-wt. per sq. ft. (2) 233,280 lb.-wt. per sq. yd.
14. $13 \cdot 74$ lb.-wt. per sq. inch.
15. (1) $2033 \cdot 6$ grms.-wt. per sq. cm. (2) 6168 grms.-wt. per sq. cm.
16. $46 \cdot 08$ ft.
17. 1000 kilogrammes weight.
18. 73,500 c. cm.
19. 2000 lb.-wt. per sq. foot.
20. $41 \cdot 3$ lb.-wt. per sq. ft.
21. $5 \cdot 63$ lb.-wt. per sq. inch.
22. $11 \cdot 41$ inches.
24. $69 \cdot 12$ ft.
26. 3373 lb.-wt.
27. $11 \cdot 76$ lb.-wt. per sq. inch.
28. $45 \cdot 5$ grms.-wt. per sq. cm.
29. $\cdot 192$ sq. ft.
30. (i) Upward thrust on top of barrel $=208\frac{1}{3}$ lb.-wt. (ii) volume of water $=144$ cubic inches.

CHAPTER IV. (Page 93.)

2. (a) $4 \cdot 948$ lb.-wt. (b) 61,500 tons-wt.
3. $\dfrac{\text{thrust on base with vertex upwards}}{\text{vertical thrust on curved surface, vertex downwards}} = \dfrac{3}{\sqrt{5}}$.
5. $1 \cdot 0167$ ft.
6. (a) $\dfrac{\text{thrust on base of large cistern}}{\text{thrust on base of small cistern}} = 8$.

   (b) $\dfrac{\text{thrust on vertical sides of large cistern}}{\text{thrust on vertical sides of small cistern}} = 16$.
7. 29 lb.-wt.
8. $283\frac{1}{3}$ lb.-wt.
9. A force equal to the weight of $562\frac{1}{2}$ lb. applied to the centre of the lower edge of the face.
10. 1875 lb.-wt.
11. 50,625 lb.-wt.
12. $\dfrac{\text{thrust on side}}{\text{thrust on bottom}} = \frac{5}{28}$.
13. $351,562\frac{1}{2}$ lb.-wt.
14. $1,001,953\frac{1}{8}$ lb.-wt.
15. $25 \cdot 323$ ft.
16. $2273\frac{1}{4}$ lb.-wt.
17. 46,256 lb.-wt.
20. $\dfrac{\text{Force on large plate}}{\text{Force on small plate}} = \frac{25}{9}$.

CHAPTER V. (Page 114.)

1. 25·9 cubic inches. 2. 87·11 grms.
3. Resultant thrust of water $= 41\frac{2}{3}$ kilogrammes-wt. Acceleration $= 3106\frac{1}{2}$ cm. per sec. per sec.
4. (1) 100 grms.-wt. (2) 28·66 grms.-wt. (3) 925 grms.-wt.
5. ·57 of its volume.
6. Volume of cork = 15·8 times the volume of the iron.
7. 22·44 grms. 8. 6·257 grms. 9. 19,000 c. ft.
10. 65·8 c. inches. 11. 45 c. cm. 12. 0·97.
13. (1) ·5454 lb. (2) ·3939 lb. (3) ·425 lb. 14. 3·07 grms.
15. Wt. of most dense body = 9 times the wt. of least dense body.
16. Mean spec. gravity = 0·9987. Volume = 3875·72 c. inches.
17. ·55 of the weight.
18. ·537 of the volume of the iron is in mercury, and ·463 of the vol. of the iron is in water.
19. Spec. gravity of wood = ·533. Spec. gravity of cork = ·25. Spec. gravity of ice = ·919. Spec. gravity of oak = ·75.
20. ·726 inches.
21. Volume of iron = 36·4 c. cm. Density of iron = 7·56 grms. per c. cm.
22. ·48 cubic inches. 23. ·45 inches. 24. Sp. gr. = ·5.
25. 2·34 cm. in water. 2·66 cm. in mercury. 27. 93·75 cm.
28. (1) Pressure required = 5 lb.-wt. (2) Wt. of metal $= 6\frac{1}{4}$ lb.
30. (1) Floats in ordinary water with ·0064 inches above the surface.
(2) ,, sea-water ,, ·108 ,, ,, ,,
31. 5 lb.-wt. 33. 20 c. cm. 36. 89·8 ft.
37. Mass of iron = 1·66 grms. Mass of wax = 34·34 grms.
38. 0·93. 39. 277·0 c. inches.

CHAPTER VI. (Page 137.)

1. (1) 2·637. (2) 11·37. (3) 2·68. (4) 17·57.
2. (1) ·240. (2) ·8535. (3) ·530. (4) ·601.
3. (1) 1·0714. (2) 1·200. (3) 1·276. (4) 1·305.
4. Spec. gravity = 3·57. Volume = 7·69 c. cm. 5. 4·04.
6. 1·069. 7. 43·79 grms. 8. 2·476.
9. (1) 1·064. (2) 0·920. (3) 1·032.
10. 2·077. 11. 2·768.
12. Mass of silver = ·997 times the mass of the gold. 13. $1\frac{1}{11}$.
14. ·309 of the volume of the mixture is alcohol.

15. 82·9 cm. 16. 6·36 inches. 18. 12·288.
19. Specific gravity = 3. Volume = 1 c. cm. 20. ·923.
22. 8·88. 23. 1·5. 24. ·80.
25. ·787. 26. 13·92 grms. 27. 1·44.
28. 1·48 c. cm. 30. 2·26.
31. Mercury falls 1·99 cm. in one leg, and rises 1·99 cm. in the other. 65·9 cm. of oil are required. 32. 27·2 inches.

CHAPTER VII. (Page 166.)

1. (1) 3800 c. cm. (2) 2992 c. cm. (3) 3839 c. cm. (4) 47·1 c. cm.
2. 39·1 litres. 3. 18·12 kilogrammes.
4. 340·5 kilogrammes. 5. 144·16 grms.-wt. per square cm.
6. Add 13·36 mm. 7. 4·05 mm. 8. 75·394 grms.
9. Pressure of hydrogen = 14·4 times the pressure of the air.
10. (1) 749·54 mm. (2) 27·445 inches. (3) 21·176 inches.
11. (1) 783 c. cm. (2) 306·1 c. cm. (3) 414·7 cubic inches.
12. 26·48 inches. 13. 2·401 litres.
14. 24·73 inches of mercury. 15. 8752 yards.
16. Increases by 10 cubic feet.
17. 1·782 inches in diameter. 18. ·0015 cubic inches.
19. Diminished to $\frac{1}{20}$th of its former value. 20. 2·85 cubic ft.
21. $\frac{4}{7}$th of the air escapes, or, in other words, the escaped air occupies a volume of $85\frac{1}{3}$ cubic inches.
22. 60 lb.-wt. 23. 8932 yards. 24. ·2533 grms.
27. Weight on mountain = $\frac{4}{5}$th weight at sea-level.
29. 30 inches of mercury.
30. Pressure of the inside air is less than the pressure of the outside air by the pressure of the column of water left in the bottle.
32. 8408 metres. 33. 6181 ft. 35. 30·5 inches.

CHAPTER VIII. (Page 204.)

4. (1) 30 inches. (2) 513·2 inches. 6. 64·4 lb.-wt.
7. 4 strokes. 8. 8·1 lb.-wt. 9. $47{,}142\frac{6}{7}$ lb.-wt.
10. 37,500 ft.-lb. per minute.
11. (1) Force required = 13·635 lb.-wt. (2) Work done = 24,543 ft.-lb. (3) Wt. of water raised = 981·72 lb.
12. 10·2 strokes.

13. 981·75 lb.-wt. during the back stroke and 3927 lb.-wt. during the forward stroke.

14. 15,590 ft.-lb. 15. 304·5 inches.

16. (1) $\frac{243}{1024} \times$ original pressure. (2) $\frac{59,049}{1,048,576} \times$ original pressure.

(3) $\frac{14,348,907}{1,073,741,824} \times$ original pressure.

17. $\frac{3}{8}$th of a stroke. 18. 16 strokes.

19. When the piston in its upward stroke has traversed $\frac{7}{18}$ of the barrel.

20. $\frac{1}{256}$th of the atmospheric density.

21. Volume of receiver = 4 times that of the barrel.

22. 167 strokes. 24. 5 strokes.

26. $\frac{1}{1331}$ of the atmospheric density. 27. 10 strokes.

28. ·2727 cubic feet. 30. $21\frac{7}{11}$ ft. 31. 27·16 ft.

32. Depth of top of bell = 3 inches.

33. (1) 1·155 V. (2) ·311 V. Where V = volume of the bell.

34. Half way up the cylinder. 35. 300 cubic feet.

36. Until its depth be doubled (neglecting the length of the bell).

37. Top of the bell is $\left\{\rho(h'-h) - \frac{h \cdot a}{h'}\right\}$ ft. below the surface.

38. Weight raised = 1000 tons. Number of strokes required = 12,000.

39. $1\frac{1}{18}$ sq. feet. 40. $\frac{1}{4}$ ton weight.

41. Weight raised = 2800 lb. Number of strokes = 240.

42. ·426 inches.

43. Load required = 14,137·2 lb. Head of water = 1152 feet.

44. 191 lb. per sq. inch. 45. ·5 grms. per c. cm.

46. $23\frac{1}{3}$ c. cm. 47. 102·3 lb.-wt.

48. 60 lb. per sq. inch. 49. 150 lb. per sq. inch.

50. (1) Volume raised = 19,161 c. feet. (2) Reduction of volume = 2903 c. ft.

INDEX TO DYNAMICS.

The references are to the pages.

INDEX TO STATICS.

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INDEX TO HYDROSTATICS.

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